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USAGE OF SPECIAL FINITE ELEMENTS FOR SOLUTION OF FRACTURE MECHANICS PROBLEMS

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Summary. *The paper suggests the usage of special modified finite elements, which account for the square-root singularity of stress and strain fields at the crack top and allows high-precision evaluation of stress intensity factors. These elements are introduced into the programs of finite element method for the analysis of plate problems of termoelasticity of anisotropic solids containing cracks. The comparison is done between the known analytic solutions and the results obtained using special finite elements and modified finite elements. High accuracy of the results obtained with the usage of modified special finite elements has been proved.*

Key words: *fracture mechanics, stress intensity factors, termoelasticity, crack top, modified finite elements.*

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Problem setting and publications overview. The problem of the destruction of materials and structures is one of the important problems of the mechanics of the deformable solid body. Recently, due to insufficient reliable assessment in predicting the emergence and spread of cracks in modern designs interest in the study of these processes has increased [1]. However, despite the extensive information about various destruction phenomena, its mechanism is not fully known and there is not a large number of experimental studies available, particularly concerning the destruction of anisotropic bodies. This is due to the complexity and high cost of such experiments, and the inability to examine the processes of emergence and further development of cracks.

The most effective method to be used for this analysis is the method of numerical experiment. Numerical methods study design of structures with and without cracks of varying geometry of individual items and change in terms of loading in the current node. This method of analysis has become possible due to wide practical use of numerical methods for calculating the stress-strain state (SSS) designs, including finite element method (FEM).

There are several approaches to build finite element models of design with such damages as cracks. The first partition uses the entire study area with conventional finite element mesh with a significant increase towards the top of damage. The second is to use special elements surrounding the top of the damage and provide an opportunity to consider features of strain distribution near the damage [2 – 4].

Methods for determining the durability of cracked bodies requiring prior calculation of stress intensity factors coefficients (SIC), which are usually prior unknown. In addition, the nature of stress changes is given (root feature), so it is difficult to get without taking into account theoretically grounded concordant results. The most correct approach is to use special items which model singularity of stresses and strains in the crack top (singular elements). These elements reflect features of SSS in the vicinity of the crack peak.

These elements are called special as in calculating the stiffness matrix different from the usual tool for moving shapes containing proportional member. They also differ from conventional FE because their intermediate nodes are shifted by a quarter-length side towards the crack top. These elements may have features like $O\left(r^{-1/2}\right)$ for strains, quite well describe

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the change of stresses and displacements at the top of the crack, are fully compatible with conventional quadratic elements and reflect the movement of the body as a whole. Moreover, theorems on convergence solution close to the exact as for common elements remain valid.

In this paper, the analysis of stress-strain state in the vicinity of the crack top in metal plate using analytical models and methods based on the use of special finite element (FE), describing the features of the stress field at the crack top has been conducted.

Based on modern concepts of SSS, in case of tension cracks and deformations in the vicinity of the top the following general correlations are described [5]:

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta), \ \varepsilon_{ij} = \frac{K}{\sqrt{2\pi r}} q_{ij}(\theta) \tag{1}$$

where σ_{ij}, ε_{ij} – strain tensor components;
K – stress intensity factor, defined as K_I, K_{II}, K_{III} для for appropriate modes of destruction;
r, θ – polar coordinates beginning in the crack top, which is located along the axis x;
$f_{ij}(\theta)$, $q_{ij}(\theta)$ – universal normalized functions.

Displacement should look like $u_i = \frac{K}{G}\sqrt{\frac{r}{2\pi}} F_i(\theta)$,

where G – shear modulus; $F_i(\theta)$ – universal normalized function.

To calculate with the displacement method we should further define the location of the crack top. In this case, based on the asymptotic formulas obtained by Irwin, SSS is determined in the vicinity of the crack top and SIC:

first mode (normal lead) singular stress field near the crack has the form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos(\frac{\theta}{2}) \begin{Bmatrix} 1 - \sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) \\ 1 + \sin(\theta/2)\sin(3\theta/2) \end{Bmatrix} \tag{2}$$

and the corresponding displacements

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_I}{2G}\left(\frac{r}{2\pi}\right)^{1/2} \begin{Bmatrix} \cos(\theta/2)\left[k - 1 + 2\sin^2(\theta/2)\right] \\ \sin(\theta/2)\left[k + 1 - 2\cos^2(\theta/2)\right] \end{Bmatrix} \tag{3}$$

where $k = 3 - 4\nu$ for plane strain, $k = (3-\nu)/(1+\nu)$ for plane stress
for the second mode (shift) singular stress field has the form:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{Bmatrix} = \frac{K_{II}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\theta/2)\left[2 + \cos(\theta/2)\cos(3\theta/2)\right] \\ \cos(\theta/2)\left[1 - \sin(\theta/2)\sin(3\theta/2)\right] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{Bmatrix} \tag{4}$$

and corresponding displacements

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{K_{II}}{2G}\left(\frac{r}{2\pi}\right)^{1/2} \begin{Bmatrix} \sin(\theta/2)\left[k+1+2\cos^2(\theta/2)\right] \\ -\cos(\theta/2)\left[k-1-2\sin^2(\theta/2)\right] \end{Bmatrix} \tag{5}$$

third-mode (longitudinal shear crack) singular stress field is:

$$\begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \frac{K_{III}}{\sqrt{2\pi r}} \begin{Bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix} \tag{6}$$

and corresponding displacements

$$\{u_3\} = 2\frac{K_{III}}{G}\left(\frac{r}{2\pi}\right)^{1/2} \sin(\theta/2) \tag{7}$$

***J* – integral**

Recognized basic parameters of fracture mechanics and SIC are integral Cherepanov – Rice's (J – integral), which are calculated based on direct and energetic techniques with synonymous connection within linear fracture mechanics:

$$J = \int_G \left(W dy - T \frac{\partial u}{\partial x} ds \right) \tag{8}$$

where *G* – closed loop that goes through anti-clockwise, which limits certain region in the vicinity of the crack top;

T – stress vector perpendicular to the path *G* $T_i = \sigma_{ij} n_j$;

u – displacement towards the axis *x*;

ds – circuit element *G*;

W – the energy of deformation $W = W(x, y) = W(\varepsilon) = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij}$;

σ_{ij}, ε_{ij} – components of the stress tensor.

The surface to calculate the J – integral in the neighborhood of an arbitrary point field cracks will consist of contour (F_k) and two side components (F_1 та F_2) [6]:

$$J = \frac{1}{\Delta}\left(J_{F_K} + J_{F_1} + J_{F_2} \right) \tag{9}$$

In casde of a temperature field, invariant integral expression is the following [6]:

$$J^* = J + \int_S \alpha \sigma_{ij} \delta_{ij} \frac{\partial T}{\partial x} dS \frac{1}{\Delta} \tag{10}$$

where *J* – integral form (8).

Connection SIC of magnitude of *J* – integral in linear deformation conditions is determined by the formula [7]:

$$J = \frac{kK_1^2}{E} \tag{11}$$

where: $k=1$ under the plane stress conditions i $k=1-\nu^2$ for plane strain; E – Young's modulus.

At a constant temperature T in the vicinity of the crack connection between displacement and K_1 describes dependency [6]:

$$u_i = \frac{K_I}{2G}\sqrt{\frac{r}{2\pi}}F(\theta) + \bar{\alpha}kTy^i \tag{12}$$

where G – shear modulus;

$\bar{\alpha} = \alpha$ – linear expansion coefficient in the case of plane stress,

$\bar{\alpha} = \alpha(1+\nu)$ for plane strain:

$F_1(\theta) = \sin\frac{\theta}{2}\left(k+1-2\cos^2\frac{\theta}{2}\right)$; $F_2(\theta) = \sin\frac{\theta}{2}\left(k-1+2\sin^2\frac{\theta}{2}\right)$

where $k=3-4\nu$ for plane strain, $k=(3-\nu)/(1+\nu)$ for plane stress.

Singular finite element. In numerical modeling of processes of destruction for the exact value of stress intensity important factor is to determine local displacement field efforts and cracks in its top. To construct the displacement field at the crack top and its geometry *Williams M.L.* [7] used the dependence

$$\begin{gathered} u_k = (r,\theta) = a_k + b_k(\theta)r^{\frac{1}{2}} + c_k(\theta)r + d_k(\theta)r^{\frac{3}{2}} + L \\ k = 1, 2, ... \end{gathered} \tag{13}$$

where r distance to the crack top θ corner crack propagation as shown in Figure 1.

So crack width is defined as

$$\begin{gathered} \Delta u_k = \Delta b_k(\theta)r^{\frac{1}{2}} + \Delta c_k(\theta)r + \Delta d_k(\theta)r^{\frac{3}{2}} + L \\ k = 1, 2, ... \end{gathered} \tag{14}$$

In the finite-element modeling to determine SIC cracks *Henshell R.D.* and *Shaw K.G.* [8] and *Barosum R.S.* [9] suggested the use of a standard approach, combined with displacement of finite element nodes at the top of a quarter crack length element.

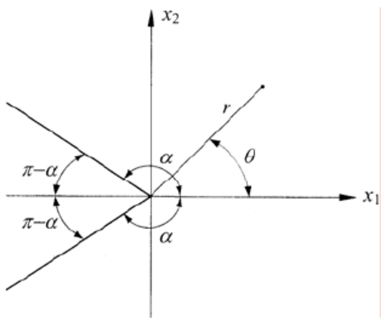

Figure 1. Symbols for the system of coordinates (x_1, x_2) and (r, θ) crack geometry

Use of √r – singular element is important because when you reach a boundary change, its modifications occurs. Researchers *Gray L.J.* and *Paulino G.H.* have shown that regardless of geometry or boundary conditions, linear coefficients according to *Williams M.L.* have the compliance:

$$c_k(\pi) = c_k(-\pi) \tag{15}$$

where $\Delta u_k(r)$ describes the movement of cracks. Since the linear factor disappears from the expression determining crack width along its distribution

$$\Delta u_k(r) = u_k(r,\pi) - u_k(r,-\pi) \tag{16}$$

Standard √r – singular element does not comply with the limits that were given in equations (15) and received in equations (16). So to get an accurate assessment of SSS in the vicinity of the crack top linear rate $\Delta u_k(r)$ should approximate to zero. Obtaining important and interesting interpretation of analytical results in fracture mechanics of solids is still unknown.

Modified singular finite element. Two-dimensional √r – a singular element that is shown in Figure 2, is based on the use of six-nodal triangular element. To $t \in [0,1]$ crack at the top of the tool shapes $t = 0$ for elements along the crack boundary (corresponding units 1, 2 and 4) takes the form

$$\begin{aligned} \varphi_1(t) &= (1-t)(1-2t) \\ \varphi_2(t) &= t(2t-1) \\ \varphi_4(t) &= 4t(1-t) \end{aligned} \tag{17}$$

Since at the crack top $(\Delta u_1^1, \Delta u_2^1) = (0,0)$, taking into account boundary constraints crack opening field (use of units 1, 2 and 4) and Δu_k takes the form:

$$\begin{aligned} &\Gamma(t) \equiv \left[(x,y) = x_1\varphi_1(t) + x_2\varphi_2(t) + x_4\varphi_4(t),\, y_1\varphi_1(t) + y_2\varphi_2(t) + y_4\varphi_4(t)\right] \\ &\Delta u_1(t) \equiv \Delta u_1^2\varphi_2(t) + \Delta u_1^4\varphi_4(t) = -\left(\Delta u_1^2 - 4\Delta u_1^4\right)t + \left(2\Delta u_1^2 - 4\Delta u_1^4\right)t^2 \\ &\Delta u_2(t) \equiv \Delta u_2^2\varphi_2(t) + \Delta u_2^4\varphi_4(t) = -\left(\Delta u_2^2 - 4\Delta u_2^4\right)t + \left(2\Delta u_2^2 - 4\Delta u_2^4\right)t^2 \end{aligned} \tag{18}$$

where (x_1, y_1), (x_2, y_2) та (x_4, y_4) coordinates of corresponding node elements – 1, 2 and 4; Δu_k^j – nodal values of crack opening in the $j\ th$ – node.

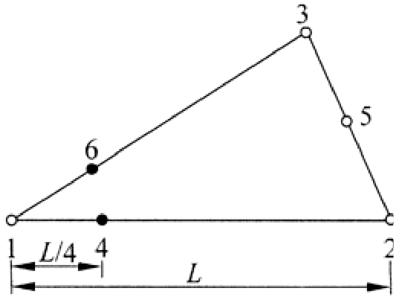

Figure 2. √r – square-root singular element

As shown earlier [7, 8] when you change the coordinates of a quarter the length of the item at the top of the crack, the value *t* is equal $(r/L)^{1/2}$, where L – the distance between nodes

(x_1, y_1) and (x_2, y_2). Consequently, a member of the first order Δu_k, which determines t, characterizes root feature distance $(r/L)^{1/2}$. However, the following describes a member t^2 r/L. The resulting value makes it possible to more accurately determine SIC. Thus the function forms for units 2 and 4 $\overline{\varphi_2}(t)$, $\overline{\varphi_4}(t)$, which are shown in equation (19), store the parameters $t = \sqrt{r}$ and carry out a substitution of $(r/L)^{1/2}$ by $(r/L)^{3/2}$.

$$\overline{\varphi_2}(t) = t(2t-1) + 2t(1-t)(1-2t)/3 = \frac{1}{3}(4t^3 - t)$$
$$\overline{\varphi_4}(t) = 4t(1-t) - 4t(1-t)(1-2t)/3 = -\frac{8}{3}(t^3 - t) \tag{19}$$

Modification of units 2 and 4 reveals the root feature $t^2 = r$ without losing the degree of interpolation, ie

$$\overline{\varphi_2}(0) = 0,\ \overline{\varphi_2}(1/2) = 0,\ \overline{\varphi_2}(1) = 1;\quad \overline{\varphi_4}(0) = 0,\ \overline{\varphi_4}(1/2) = 1,\ \overline{\varphi_4}(1) = 0 \tag{20}$$

In addition, as shown in Fig. 3, the modification (replacement) remains unchanged for form functions $\overline{\varphi_2}(t)$, $\overline{\varphi_4}(t)$ that are used in the calculation of crack opening in the equation (18).

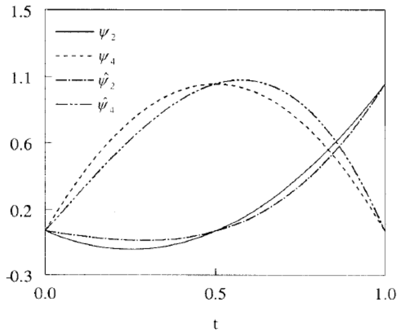

Figure 3. Standard $\varphi_2(t)$, $\varphi_4(t)$ and modified $\overline{\varphi_2}(t)$, $\overline{\varphi_4}(t)$ shape functions

Calculation of SIC will hold correlation method using √r – singular elements (standard and modified). However, it should be noted that the assessment of quality √r – singular element method is done using correlation movement. The main dependence for SIC for joint action of several events, that is, K_I and K_2 with the use of correlation bias:

$$K_I = \frac{G}{k+1} \lim_{r \to 0} \sqrt{\frac{2\pi}{r}} \Delta u_2$$
$$K_{II} = \frac{G}{k+1} \lim_{r \to 0} \sqrt{\frac{2\pi}{r}} \Delta u_1 \tag{21}$$

where $\Delta u_k(r)$ – crack opening in a system of coordinates that matches the top cracks;
G, ν – shear modulus and Poisson's ratio respectively;
$k = 3 - 4\nu$ for plane strain, $k = (3-\nu)/(1+\nu)$ for plane stress.

Using a modified √r – a singular element to the equation (19) we obtain

$$\Delta u_k = \Delta u_k^2 \bar{\varphi}_2(t) + \Delta u_k^4 \bar{\varphi}_4(t) = \frac{4}{3}\left(\Delta u_k^2 - 2\Delta u_k^4\right)t^3 + \frac{1}{3}\left(8\Delta u_k^4 - \Delta u_k^2\right) \tag{22}$$

Using (22) and (21) taking into account conditions $t = \sqrt{r/L}$

$$K_I = \frac{G}{3(k+1)} \lim_{r\to 0} \sqrt{\frac{2\pi}{L}} \left(8\Delta u_2^4 - \Delta u_2^2\right)$$
$$K_{II} = \frac{G}{3(k+1)} \lim_{r\to 0} \sqrt{\frac{2\pi}{r}} \left(8\Delta u_1^4 - \Delta u_1^2\right) \tag{23}$$

Numerical example. We consider boundless environment (square with sides that are ten times bigger than the length of the crack) with a straight insulated crack of length $2a$ and inclined to the axis Ox_1 at an angle γ. At infinity uniform heat flux effect with vector components of such density: $h_1^\infty = 0$, $h_2^\infty = h^\infty$. Tensions disappear at infinity.

Further we explore the stress intensity factor for the crack plane stress when the material is isotropic medium (Poisson's ratio 0.25) or anisotropic with the following properties: $E_{11} =$ 55 GPa, $E_{22} = 21$GPa, $\nu_{12} = 0.25$, $G_{12} = 9,7$ GPa, $\alpha_{11} = 6,3 \cdot 10^{-6}$ K^{-1}, $\alpha_{22} = 2 \cdot 10^{-5}$ K^{-1}, $k_{11}/k_{22} = 3,46/0,35$. We model sotropic material with weakly anisotropic perturbations from 0.1% shear modulus.

Table. 1 presents values normalized SIC at the crack left top (the right side of CIN with opposite sign), depending on its angle of inclination compared to the data of the analytical solution. Normalization factor is $K_0 = a\sqrt{\pi a} E_{11} \alpha_{11} h^\infty / k_{11}$. The model uses 400 finite elements.

Table. 1 shows good consistency of numerical results with analytical data in the case of isotropic and anisotropic material. The relative error does not exceed 0.2% for the modified special items that certifies the authenticity of the results. Conventional special items give a little more calculation error.

Table № 1.
SIC angled crack in an infinite plate

γ, град	0	30	45	60	75
Isotropic material					
K_{II}/K_0, точно	0,2500	0,2165	0,1768	0,1250	0,0647
K_{II}/K_0, МСЕ	0,2515	0,2210	0,1797	0,1284	0,0689
K_{II}/K_0, МСЕ, мод. елем.	0,2505	0,2169	0,1771	0,1253	0,0648
Anisotropic material					
K_I/K_0, точно	0,0000	0,3279	0,3091	0,1893	0,0566
K_I/K_0, МСЕ	0,0000	0,3296	0,3105	0,1916	0,0588
K_I/K_0, МСЕ, мод. елем.	0,0000	0,3284	0,3097	0,1896	0,0567
K_{II}/K_0, точно	0,3657	0,5060	0,5677	0,5107	0,3058
K_{II}/K_0, МСЕ	0,3723	0,5139	0,5784	0,5126	0,3103
K_{II}/K_0, МСЕ, мод. елем.	0,3664	0,5069	0,5687	0,5116	0,3063

Conclusions. The proposed method of determining the SIC with modified special finite elements can be used as a basis for the development of algorithms and numerical solution schemes corresponding boundary problems using variational methods. The resulting value is the source for the iterative construction of two-dimensional and one-dimensional mathematical models of thermomechanics elements of thin-walled structures. The resulting finite element method found good consistency of research results with known data in the study of structurally heterogeneous anisotropic bodies with cracks. Thus the proposed algorithms have the potential for implementation into application software packages of structural engineering calculation of heterogeneous anisotropic bodies.

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УДК 539.3

ВИКОРИСТАННЯ СПЕЦІАЛЬНИХ ЕЛЕМЕНТІВ МСЕ ДО РОЗВ'ЯЗУВАННЯ ЗАДАЧ МЕХАНІКИ РУЙНУВАННЯ

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Резюме. *Запропоновано використовувати спеціальні модифіковані скінченні елементи, що враховують кореневу особливість поля напружень та деформацій у вершині тріщини й дають можливість високоточного обчислення коефіцієнтів інтенсивності напружень. Ці елементи введено до комплексу програм методу скінченних елементів для аналізу плоских задач термопружності анізотропних тіл із тріщинами. Здійснено порівняння відомих аналітичних даних із результатами розрахунків із використанням спеціальних та модифікованих скінченних елементів. Засвідчено високу точність розрахунків, здійснених із застосуванням модифікованих спеціальних скінченних елементів.*

Ключові слова: *механіка руйнування, коефіцієнт інтенсивності напружень, термопружність, вершина тріщини, модифіковані скінченні елементи.*

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