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SOLUTION OF THE INVERSE PROBLEM OF VIBRATIONS OF A HETEROGENEOUS ROD

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Summary. *As an example of a limited boundary-value inverse problem of vibration theory, a solution of definition of rod elastic properties heterogeneity problem in case of longitudinal vibrations is given. It is considered that density of the rod material is constant. Geometrical dimensions of the rod are given, and heterogeneity to be defined is located in neighborhood of some cross-section of the rod. It is also considered that values of the first ten or twenty natural frequencies of longitudinal vibrations of the rod with heterogeneity are known. Proper choice of Eigen functions allowed defining the Fourier series coefficients with odd and even numbers. Given method allowed defining localized elastic properties heterogeneity defects and location of heterogeneity zones. It is shown that the results of experimental research are consistent with calculations.*

Key words: *inverse problem, low frequency tomography method, rod vibrations, heterogeneity of elastic properties.*

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Problem setting. Until recently inverse boundary problems of the theory of vibrations of thin elements with the local heterogeneity have not been the subject of systematic research in mechanics of the deformable bodies and related areas. This is largely due to the fact that the problem of this class is pretty complicated, because it belongs to the class of so-called incorrect problems, for which it is not always possible to use classical methods. On the other hand more and more of the mathematical models take the agreement and reliability just due to the achievements of the theory of inverse problems. In this regard, a separate section forming of mechanics has been observed the subject of which is the study of the general theory of the above problems, including inverse problems of spectroscopy. Unlike direct problems, which are nowadays quite well studied, solving of inverse problems is associated with overcoming of certain difficulties of analytical and computational nature, resulting, in particular, from their nonlinearity and incorrectness. Following the tradition, to the inverse we will refer, first of all, the problems of determining the geometry of the body with the known density and elastic properties on condition of known (theoretically or experimentally determined) natural vibration of frequencies. If the geometry of the body is considered to be given then the spectral methods for solving of such a problem can determine the spatial distribution of its physical properties.

Analysis of the known research results. With careful analysis of scientific works of domestic and foreign authors we can conclude that strictly mathematically inverse problem has a solution only for the problem of small transverse vibrations of the string [1]. The works of M.H.Kreyn [2], O.O.Vatulyan [3] are devoted to the review of the basic models and methods for solving of the inverse problems, and in the work of V.O.Marchenko the solution of the inverse scattering problem has been found [4]. As noted in [5], in order to build the solution of inverse problems it is necessary to have more information about the object of study, for example, the location of heterogeneity and its approximate size. It should be noted that by the American authors [6,7] the attempt to find a solution to the inverse problem of fluctuations by the so-called low-frequency tomography method (approximation of functions of finite sums of Fourier series) has been done. However, the authors were unable to determine the initial coefficients of the Fourier series of odd numbers.

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The purpose of the work. To consider the problem of determination of the heterogeneity of the elastic properties of the rod in case of the longitudinal vibrations as an example of the limited inverse boundary problem of vibrations theory one has to carry out numerical calculations and summarize their results in order to identify the size and location of the zones of heterogeneity.

Formulation of the problem. Let us consider an elastic rod with the length l with constant cross section. We will assume that at some distance x_0 from rigidly fixed end face $x=0$ there is a local area with the length 2δ, modulus of elasticity is different to that of a constant value of elastic modulus E_0 outside this area (fig.1).

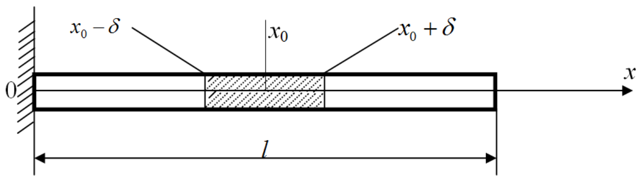

Figure 1. Rod with heterogeneity of elastic properties

We present the elastic properties of the rod as a piecewise continuous function of axial coordinates x in the following form:

$$\frac{E(x)}{E_0}=\begin{cases}1, 0\le x<x_0-\delta,\\ 1+\varepsilon, x_0-\delta<x<x_0+\delta,\\ 1, x_0+\delta<x\le l\end{cases} \qquad (\varepsilon>0) \tag{1}$$

while the density of the material of the rod is taken as constant throughout the volume $\rho = const$.

As it is known, the problem based on the own rod vibrations with heterogeneity is reduced to the integration of the generalized wave equation regarding the function of the shift of the rod points $u(x,t)$

$$\frac{\partial}{\partial x}\left[E(x)\frac{\partial u_x}{\partial x}\right]+\rho\frac{\partial^2 u_x}{\partial t^2}=0, \tag{2}$$

regarding zero boundary conditions

$$u_x\big|_{x=0}=0,\ \frac{\partial u_x}{\partial x}\Big|_{x=l}=0 \tag{3}$$

In their own vibration modes of displacement and stress they will be the harmonic functions of time, thus we put

$$u_x(x,t)=\hat{u}_x(x)\,e^{i\omega t} \tag{4}$$

After substituting (4) into equation (2) considering (1), we have:

$$\frac{d}{dx}\begin{pmatrix}1, & 0\le x<x_0-\delta & \\ 1+\varepsilon, & x_0-\delta\le x\le x_0+\delta & \frac{d\hat{u}_x}{dx} \\ 1, & x_0+\delta\le x\le x_0+\delta & \end{pmatrix}+\lambda^2\bar{u}_x=0, \tag{5}$$

where $\lambda = \frac{\omega}{c}$, $c = \sqrt{\frac{E_0}{\rho}}$, where c – is the velocity of longitudinal waves distribution in rods.

Herewith boundary conditions that have been considered (3) are transferred to the amplitude function $\hat{u}_x$ and are written as:

$$\hat{u}_x\big|_{x=0} = 0, \qquad \frac{d\hat{u}_x}{dx}\big|_{x=l} = 0 \cdot \tag{6}$$

With the entry form of equation (5) follows the obvious conclusion that out of the local area of heterogeneity ($0 \le x < x_0 - \delta,\ \ x_0 + \delta < x \le l$) amplitude functions $\hat{u}_x^{(1)}$ and $\hat{u}_x^{(3)}$ are defined in accordance with the solution of equation

$$\frac{d^2\hat{u}_x^{(i)}}{dx^2} + \lambda^2\hat{u}_x^{(i)} = 0, i = 1{,}3, \tag{7}$$

and in the area of heterogeneity – in accordance with the following:

$$\frac{d^2\hat{u}_x^2}{dx^2} + \frac{\lambda^2}{1+\varepsilon}\hat{u}_x^{(2)} = 0 \tag{8}$$

To determine the amplitude functions $\hat{u}_x^{(k)}$ $(k = 1{,}2{,}3)$ along with the boundary conditions (7) the following terms of interface solutions must be used:

$$\hat{u}_x^{(1)}\Big|_{x=x_0-\delta} = \hat{u}_x^{(2)}\Big|_{x=x_0-\delta}, \qquad \hat{u}_x^{(2)}\Big|_{x=x_0+\delta} = \hat{u}_x^{(3)}\Big|_{x=x_0+\delta}, \tag{9}$$

$$\frac{d\hat{u}_x^{(1)}}{dx}\Big|_{x=x_0-\delta} = (1+\varepsilon)\frac{d\hat{u}_x^{(2)}}{dx}\Big|_{x=x_0-\delta}, \quad (1+\varepsilon)\frac{d\hat{u}_x^{(2)}}{dx}\Big|_{x=x_0+\delta} = \frac{d\hat{u}_x^{(3)}}{dx}\Big|_{x=x_0+\delta}.$$

It should be noted, that the solutions of equations (7) and (8) are submitted by trigonometric functions of the arguments λx and $\lambda x / \sqrt{1+\varepsilon}$. Six arbitrary constants have been included into these solutions. Nontrivial zero solution of the system of six algebraic equations, that are received due to the execution of boundary conditions (7) and coupling conditions(9), is due to the condition of equality to zero of the determinant of this system of equations. Herewith, the natural frequencies of vibrations of heterogeneous rod are defined by zeros of the determinant. The following expression was received through the changes for frequency determinant:

$$\sin\left(\frac{2z\bar{\delta}}{\sqrt{1+\varepsilon}}\right)\left\{\sin[z(1-2\delta)] + \varepsilon\sin[z(\bar{x}_0 - \bar{\delta})]\cos[z(1-\bar{x}_0-\bar{\delta})]\right\} - \\ -\sqrt{1+\varepsilon}\cos[z(1-2\bar{\delta})]\cos\left(\frac{2z\bar{\delta}}{\sqrt{1+\varepsilon}}\right) = 0 \tag{10}$$

where $\lambda l = z$, $\frac{x_0}{l} = \bar{x}_0$, $\frac{\delta}{l} = \bar{\delta}$.

It should be noted that for a homogeneous rod $(\varepsilon = 0)$ the roots of equation

$$\cos \lambda l = 0 \tag{11}$$

are determined by the elementary formula

$$(\lambda l)_m^{(одн.)} = \left(m + \frac{1}{2}\right)\pi, \qquad (m = 0,1,2,\ldots) \tag{12}$$

The roots of the transcendental equation (10) were calculated by numerical method for $\varepsilon = 0.1$, $\bar{\delta} = 0.05$ and introduced in the table 1. Herewith three positions of heterogeneity have been considered: $\bar{x}_0 = \frac{1}{3}, \frac{1}{2}$ та $\frac{2}{3}$. In the first column of the table 1 the frequencies of natural vibrations of a homogeneous rod have been also recorded for comparison, which was found by formula (11).

Table № 1

Frequencies of natural vibrations of homogeneous and heterogeneous rods

	$(\lambda l)_m^{(неоднор)}$			
m	$(\lambda l)_m^{(одн.)}$	$\bar{x}_0 = 1/3$	$\bar{x}_0 = 1/2$	$\bar{x}_0 = 2/3$
0	1.571	1.5781	1.5761	1.5741
1	4.712	4.7254	4.7213	4.7173
2	7.854	7.871	7.867	7.863
3	10.996	11.014	11.010	11.006
4	14.137	14.158	14.153	14.147
5	17.279	17.304	17.299	17.291
6	20.420	20.451	20.443	20.434
7	23.562	23.601	23.588	23.579
8	26.704	26.752	26.742	26.726
9	29.845	29.913	29.899	29.872
10	32.986	33.061	33.047	33.023
11	36.128	36.215	36.204	36.172
12	39.269	39.372	39.357	39.322
13	42.412	42.531	42.521	42.478
14	45.553	45.685	45.679	45.633
15	48.695	48.852	48.833	48.786
16	51.836	52.028	52.013	51.951
17	54.977	55.230	55.192	55.126
18	58.119	58.427	58.372	58.291
19	61.261	61.639	61.588	61.451
20	64.402	64.861	64.774	64.606

For a homogeneous rod, firmly fixed to the left end face which is free from the outside pressures of the right side, normal modes of fluctuations

$$\varphi_n(x) = \sin(n + \frac{1}{2})\frac{\pi x}{l}, n = 0,1,2,\ldots \tag{13}$$

form a complete orthogonal system of functions on the interval $0 \le x \le l$. It should be noted that the system of functions is also complete on this interval

$$\psi_n(x) = \cos(n + \frac{1}{2})\frac{\pi x}{l}, n = 0,1,2,.. \qquad (14)$$

Using this, we will give approximately lump heterogeneity of elastic properties of the rod in a segment of the Fourier series

$$\frac{E(x)}{E_0} = 1 + \varepsilon \sum_{n=0}^{N} a_n \cos(n + \frac{1}{2})\frac{\pi x}{l}, \qquad N = 20$$

where the coefficients of Fourier series are given (1) as follows:

$$a_n = \frac{4}{\pi}\frac{1}{n + \frac{1}{2}} \sin[(n + \frac{1}{2})\pi \bar{\delta}]\cos[(n + \frac{1}{2})\pi \bar{x}_0], n = 0,1,2,.. \qquad (15)$$

Turning to the amplitude functions in the equation (2) and taking according to the terms of fixing the solution for amplitude displacements in a Fourier series $\hat{u}(x) = \sum_{k=0}^{\infty} A_k \sin\left(k + \frac{1}{2}\right)\frac{\pi x}{l}$, we will obtain a homogeneous system of algebraic equations

$$\left[\left(\frac{\lambda l}{\pi}\right)^2 - \left(m + \frac{1}{2}\right)^2 \left(1 + \frac{\varepsilon}{2\pi}\sum_{n=0}^{\infty}(-1)^n \alpha_{mn} a_n\right)\right] A_m -$$
$$- \varepsilon \frac{(-1)^m}{2\pi}(m + \frac{1}{2})\sum_{k=0}^{\infty}{}' (-1)^k \left[(k + \frac{1}{2})\sum_{n=0}^{\infty}(-1)^n \gamma_{mn}^{(k)} a_n\right] A_k = 0, \qquad (16)$$

where

$$\alpha_{mn} = \left[\frac{2}{n + \frac{1}{2}} - \frac{1}{n + 2m + \frac{3}{2}} - \frac{1}{n - 2m - \frac{1}{2}}\right] \qquad (17)$$

$$\gamma_{mn}^{(k)} = \left[\frac{1}{n - m + k + \frac{1}{2}} + \frac{1}{n + m - k + \frac{1}{2}} - \frac{1}{n + m + k + \frac{3}{2}} - \frac{1}{n - m - k - \frac{1}{2}}\right]$$

$$\gamma_{mn}^{(k)} = \alpha_{mn}$$

Bar in the amount shows that the parts from $k = m$ have been omitted.

The roots of the determinant of the system (16) determine the natural frequencies of vibrations of homogeneous rod. Substituting the values a_n , which have been found by the formula (15) in (16) considering (17), for $N = 20$, $k = 0,1,...,20$, $m = 0,1,...,20$ we will obtain numerical values of spectra of natural frequencies of vibrations of the heterogeneous rod.

Herewith off diagonal elements of the determinant of equations system (16) can be ignored. These values are given in table 2.

Table № 2

Values of frequency spectra of longitudinal vibrations of a rod. Approximate solution.

	$(\lambda l)_m^{(неоднор)}$		
m	$\bar{x}_0 = 1/3$	$\bar{x}_0 = 1/2$	$\bar{x}_0 = 2/3$
0	1.578	1.576	1.574
1	4.7252	4.7212	4.7171
2	7.870	7.8662	7.8622
3	11.0131	11.009	11.0055
4	14.157	14.152	14.1463
5	17.302	17.297	17.290
6	20.449	20.442	20.432
7	23.596	23.586	23.576
8	26.748	26.739	26.722
9	29.909	29.897	29.869
10	33.053	33.041	33.017
11	36.209	36.200	36.163
12	39.359	39.349	39.311
13	42.511	42.509	42.467
14	45.661	45.661	45.619
15	48.824	48.811	48.768
16	51.991	51.983	51.928
17	55.189	55.152	55.109
18	58.379	58.331	58.261
19	61.601	61.537	61.421
20	64.801	64.728	64.569

The problem of longitudinal vibrations of the rod with the heterogeneity while researching using the method of low-frequency tomography is reduced to the renovation of heterogeneity by comparing the frequency of spectra of vibrations of the homogeneous rod, which have been found using the equation (11) and experimental problems for the frequency of natural vibrations, which are measured on the rod with a given defect. Latest we will consider as "experimental", as they correspond to the data, which are usually obtained in the works of the nondestructive control of the objects. In this case the roots of characteristic equation are taken as the experimental data (10). Comparing these frequency spectra of the direct problem, we will obtain a system of algebraic equations relative to the Fourier coefficients of heterogeneity:

$$\left(m+\frac{1}{2}\right)^2 \frac{\varepsilon}{2\pi}\sum_{n=0}^{N}(-1)^n \alpha_{mn} a_n = \left(\frac{\lambda m}{\pi}\right)^{2(експ.)} - \left(\frac{\lambda m}{\pi}\right)^{2(одн.)} \qquad (18)$$

$m = 0,1,2,\ldots,20;\ \ N = 20$.

It should be noted that the written above system of algebraic equations (18) is nondegenerate and has a unique solution.

Analysis of numerical results. The numerical values of the restored coefficients of the Fourier series have been found a_n (with even and odd numbers) for the relative sizes $\varepsilon = 0.1$, $\bar{\delta} = 0.05$ and three positions of heterogeneity: $\bar{x}_0 = 1/3;1/2;2/3$.

For example, for $\bar{x}_0 = 1/3$ fig. 2 shows a graph of the defect recovery.

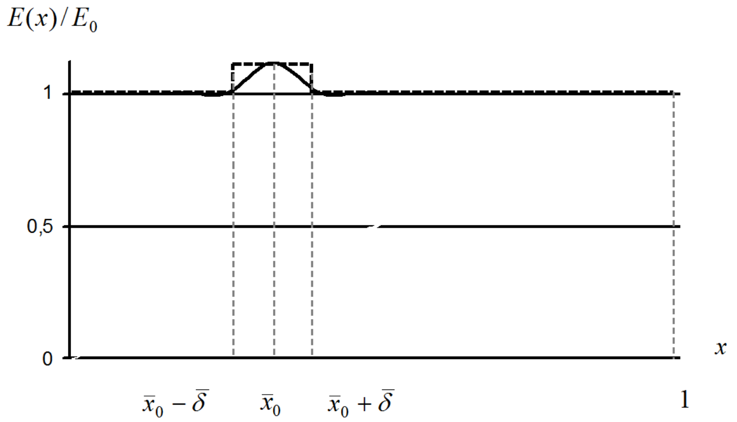

Figure 2. Tomographic recovery of heterogeneity

Thus on the graph by means of the curve with the step the true dependence of heterogeneity is depicted (formula 1), and by means of the a smooth curve – tomographic recovery of heterogeneity taking into account the restored coefficients of the Fourier series a_n, which have been found from the solution of the system of algebraic equations (18). It should be noted that the defect value which have been found by means of the reviewed method agrees with a certain heterogeneity (1).

Research results. It should be noted that the chosen approach and results let us determine the nature and location of the zones of heterogeneity. The results can be used in modern tasks of construction, as well as in the fields, as non-destructive testing, as well as finding the solution of inverse problems of vibrations.

Conclusions. For the rod with the specified conditions of fixing and heterogeneity of elastic properties, the mathematical bases of the method of low-frequency tomography are provided, which is based on the representation of the local heterogeneity segment of Fourier series. Exact and approximate solutions of the direct and inverse problems are built in case, when heterogeneity of elastic properties of the material of the rod is given in a segment of the Fourier series, but not with their own forms of the original problem, but in the full orthogonal system of union functions. Unlike the works of American authors [6,7], by means of more successful choice of their own functions, according to which there is a division of heterogeneity of elastic properties of the material rod, that are considered, it was possible to determine the initial coefficients of the Fourier series not only with even, but with odd numbers. The obtained results accurately coincided with theoretical calculations. This allows us to state that low-frequency tomography methods let us determine the location and nature of heterogeneity in case when only few first normal modes of vibrations are known. The graph of the restoration of the defect is has been provided.

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УДК 539.3

РОЗВ’ЯЗОК ОБЕРНЕНОЇ ЗАДАЧІ КОЛИВАНЬ НЕОДНОРІДНОГО СТЕРЖНЯ

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Резюме. *Як приклад обмеженої оберненої крайової задачі теорії коливань наведено розв’язок задачі про визначення неоднорідності пружних властивостей стержня у випадку повздовжних коливань. Вважається, що густина матеріалу стержня стала по всьому об’єму. Геометричні розміри стержня задані, а неоднорідність, яку необхідно визначити локалізована в околі деякого поперечного перерізу стержня. При цьому вважаються відомими значення перших десяти-двадцяти власних частот повздовжних коливань стержня з неоднорідністю. Вдалий вибір власних функцій дозволив визначити коефіцієнти ряду Фур’є з парними та непарними номерами. Представленим методом вдалося визначити локалізовані дефекти неоднорідності пружних властивостей та місце розташування зони неоднорідностей. Показано, що результати експериментальних досліджень добре узгоджуються з розрахунками.*

Ключові слова: *обернена задача, метод низькочастотної томографії, коливання стержня, неоднорідність пружних властивостей.*

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