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# MODEL OF TRANSVERSE-TRANSVERSE TYPE PIEZOELECTRIC TRANSFORMER

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*Summary. The mathematical model of a piezoelectric transformer of the transverse-transverse type and describes the method of its construction has been presented. Although mathematical modeling programs for piezoelectric devices can achieve any predetermined modeling accuracy, the simulation results cannot be directly used in the development of electronic equipment, because the programs are not integrated with CADs, for this reason most often in calculations and in modeling circuits based on piezotransformers, the simplest equivalent circuit is used. But its adequately reflects currents and voltages in the piezotransformer circuit only in the vicinity of the operating resonant frequency. The proposed model is based on a one-dimensional approximation of the equations of state and dynamics of the piezoelectric medium for flat plates of constant thickness and width, which is obtained from a three-dimensional system of equations by averaging the width and thickness. While the usual approximate model often allows to model a piezotransformer with two pairs of electrodes and only in the vicinity of one resonant frequency, the model constructed in the article allows to take into account the presence of several electrodes on piezotransformer surfaces and their different relative positions on the upper and lower surfaces. 'esoplastin. Compared with the usual, the proposed model is more convenient for modeling by means of circuit modeling systems. In the developed model, the piezotransformer is represented as a set of interconnected sections that carry one pair or several pairs of electrodes on the surfaces. Also, in contrast to the usual, the proposed model allows to take into account the presence of several resonant frequencies of the piezotransformer, which allows more adequate modeling of electronic equipment that uses in its structure a piezoelectric transformer of the transverse type. On the basis of the mathematical model the scheme of substitution of separate sections of the piezoelectric transformer is constructed and formulas for calculation of parameters of elements of the scheme are given. In in the article as example the implementation of the developed model in the computer-aided design system MicroCAP has been showed.*

*Key words: mathematical model, piezotransformer, piezoelectric element, MicroCAP.*

**Statement of the problem.** One of the areas of the development of high-frequency electronic starting devices (HFESD) is the replacement of electromagnetic components (transformer, inductor), which are their components, with piezoelectric transformers (PT) [1, 3, 4, 5], resembling transformers with scattering by their characteristics, but have much smaller mass and dimensions and do not contain electromagnetic components. Therefore, the creating electronic HFESD on the basis of PT still remains relevant.

**Analysis of available solutions and results.** For modeling and calculations of piezotransformers, as a rule, the method of electromechanical analogies is used, as well as experimental and analytical methods for studying the characteristics [2]. However, today in CAD systems of electronic equipment there is a situation in which, on the one hand, the programs of mathematical modeling of piezoelectric devices can achieve any predetermined modeling accuracy, and on the other – the simulation results can not be directly used in the process development of electronic equipment. This is due to the fact that the simulation programs are not integrated with the CAD systems themselves. For this reason, the most frequently used is the simplest model, which does not take into account even the second and third harmonics of the fundamental resonance frequency of piezoelectric elements [9–12].

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**Objectives of the research** is to draw such a model which could be both acceptable by accuracy and take into account at least a few first harmonics (that, in fact, is sufficient for practical tasks) and be simple enough for the interactive change of parameters in the process of modelling radioelectronic devices and would make it possible to consider and change the parameters if piezoelectric devices without exiting the basic CAD itself.

**Basic assumptions in modeling.** To simplify the calculations, we will limit ourselves to the simplest one-dimensional model of a piezotransformer, which is quite accurate and is used to solve most practical problems. In the model, it is taken into account that the main mechanical oscillations occur along the axis of the piezoelectric element.

We assume that the forces, voltages, velocities, potentials and currents can be considered as can be represented as a Fourier transform

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) exp(j\omega t) dt \tag{1}$$

And, in all cases we will omit the brackets to simplify and reduce the record of dependences. Also, to reduce the volume, we will describe the values of Fourier – images, omitting the phrase «Fourier image». Therefore, to model the PT, we conditionally divide it into parts, each of which has a constant width, is completely covered by electrodes and has a constant direction of polarization.

**Analysis of a separate section.** Consider a section of a piezotransformer that has a polarization in thickness, with the dimensions shown in Fig. 1. Suppose that electrodes are placed on the upper and lower surfaces of the section so that the upper electrode of the $i$-th pair extends over the entire length of the upper surface of the plate and has a width $b_i$, and the lower electrode of the pair has the same placement and occupies the lower surface of the plate. We also assume that the gaps between adjacent different electrodes are close to zero, and the electrode system completely covers the entire surface of the piezoelectric element.

We also assume that external forces act on the right and left parts of the section, causing the piezoelectric element to move with velocities $V_1$ and $V_2$, respectively. As follows from the design of the piezoelectric element, it is not affected by any forces on the upper, lower and side surfaces that are perpendicular to the axes $x_3$ and $x_2$.
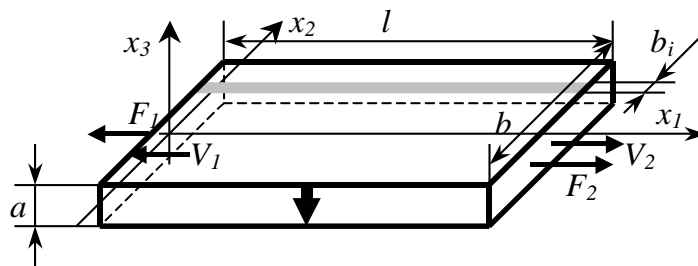


**Figure 1.** Section with transverse polarization

We use a linear model of the equation of state of the piezoelectric medium, which can be written as follows [1, 2]

$$\hat{\varepsilon}_{11} = s_{11} \hat{\sigma}_{11} + s_{12} \hat{\sigma}_{22} + s_{13} \hat{\sigma}_{33} + d_{31} \hat{E}_3 \tag{2}$$

$$\hat{D}_3 = d_{31}\,\hat{\sigma}_{11} + d_{31}\,\hat{\sigma}_{22} + d_{33}\hat{\sigma}_{33} + \epsilon_{33}\hat{E}_3, \tag{3}$$

where the symbol $\hat{\xi}$ indicates a value that depends on three coordinates. Since the speed of propagation of electromagnetic waves is much higher than the speed of propagation of mechanical oscillations in the material, a quasi-static approximation can be used for the electric field vector. Then the component of the electric field vector can be represented by the potential according to the known formula [1]

$$\hat{E}_3 = -\frac{\partial \hat{\varphi}}{\partial x_3}. \tag{4}$$

The equation of dynamics for the environment can be written as

$$\frac{\partial \hat{\sigma}_{11}}{\partial x_1} + \frac{\partial \hat{\sigma}_{12}}{\partial x_2} + \frac{\partial \hat{\sigma}_{13}}{\partial x_3} + \rho\omega^2 \hat{u}_1 = 0. \tag{5}$$

According to the zero approximation for thin beams and plates, which is widespread in solid mechanics, we will assume that the stress values $\hat{\sigma}_{22}$, $\hat{\sigma}_{33}$ are approximately 0. Note that then the stress component $\hat{\sigma}_{11}$ in the material can be determined through deformations as follows:

$$\hat{\sigma}_{11} = \frac{1}{s_{11}}\hat{\varepsilon}_{11} - \frac{d_{31}}{s_{11}}\hat{E}_3. \tag{6}$$

And the component $\hat{D}_3$ of the dielectric displacement vector can be written as

$$\hat{D}_3 = \frac{d_{31}}{s_{11}}\hat{\varepsilon}_{11} - \left(\epsilon_{33} - \frac{d_{31}^2}{s_{11}}\right)\frac{\partial \hat{\varphi}}{\partial x_3}. \tag{7}$$

Since the actual distribution of stresses and strains is not known, but, in the general case, depends on the coordinates $x_2$ and $x_3$, we enter the average values of the cross section of the plate, and will continue to use such averages that can be written without additional indications.

$$\xi_{ij} = \frac{1}{a\,b}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\hat{\xi}_{ij}dx_2 dx_3. \tag{8}$$

Carrying out the averaging over the cross section of the equation of state and taking into account the above conditions for stress, we have:

$$\sigma_{11} = \frac{\varepsilon_{11}}{s_{11}} + \frac{d_{31}}{s_{11}a\,b}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\frac{\partial \hat{\varphi}}{\partial x_3}dx_2 dx_3 = \frac{\varepsilon_{11}}{s_{11}} + \frac{d_{31}}{s_{11}}\frac{\Delta\varphi}{a}, \tag{9}$$

where $\Delta\varphi = \dfrac{1}{b} \displaystyle\int\limits_{-b/2}^{b/2} \hat\varphi\left(x_3 = \dfrac{a}{2}\right) - \hat\varphi\left(x_3 = \dfrac{a}{2}\right) dx_2$ — is the average width of the potential difference between the upper and lower electrodes. When a system of electrodes is applied on the surface so that the $i$-th upper and lower electrodes have a width $b_i$, and the potential difference between them is equal $\Delta\varphi_i$, then

$$\Delta\varphi = \frac{1}{b}\sum_i b_i \Delta\varphi_i \,. \tag{10}$$

In most practical cases, we can assume that the value of the displacement along the $x_1$ axis, averaged below the surface of the electrode, is approximately equal to the value of the movement averaged over the full cross section of the plate, i.e., is approximately equal:

$$\frac{1}{a\,b_i} \int\limits_{-a/2}^{a/2} \int\limits_{S(b_i)} \hat u_1 dx_2 dx_3 \approx \frac{1}{a\,b} \int\limits_{-a/2}^{a/2} \int\limits_{-b/2}^{b/2} \hat u_1 dx_2 dx_3 = u_1 \,, \tag{11}$$

where indication $\displaystyle\int\limits_{S(b_i)} ....dx_2$ denotes the integral by the width of $i$-th pair of electrodes. Having differentiated this expression by $x_1$, the approximated equality is obtained:

$$\varepsilon_{11} \approx \frac{1}{a\,b_i} \int\limits_{-a/2}^{a/2} \int\limits_{S(b_i)} \hat\varepsilon_{11} dx_2 dx_3 \,. \tag{12}$$

Now consider the equation of state. The component of the dielectric displacement vector, averaged over the width of the $i$-th pair of electrodes, is equal to

$$D_3^{(i)} = \frac{1}{a\,b_i} \int\limits_{-a/2}^{a/2} \int\limits_{S(b_i)} D_3 \, dx_2 dx_3 =$$

$$= \frac{d_{31}}{s_{11}} \frac{1}{a\,b_i} \int\limits_{-a/2}^{a/2} \int\limits_{S(b_i)} \hat\varepsilon_{11} dx_2 dx_3 - \left(\epsilon_{33} - \frac{d_{31}^2}{s_{11}}\right) \frac{1}{a\,b_i} \int\limits_{-a/2}^{a/2} \int\limits_{S(b_i)} \frac{\partial \hat\varphi}{\partial x_3} dx_2 dx_3 \,. \tag{13}$$

Given the above hypothesis about the distribution of the average values of deformations and integrating the last term in thickness, we have:

$$D_3^{(i)} = \frac{d_{31}}{s_{11}} \varepsilon_{11} - \left(\epsilon_{33} - \frac{d_{31}^2}{s_{11}}\right) \frac{1}{a\,b_i} \int\limits_{S(b_i)} \Delta\varphi_i \, dx_2 \,, \tag{14}$$

where $\Delta\varphi_i = \dfrac{1}{b_i} \displaystyle\int\limits_{S(b_i)} \hat\varphi\left(x_3 = \dfrac{a}{2}\right) - \hat\varphi\left(x_3 = \dfrac{a}{2}\right) dx_2$ — the value of the potential difference between the upper and lower $i$-th electrodes. Since the potential within the electrodes is not variable, the expression can be simplified and further obtained:

$$D_3^{(i)} = \frac{d_{31}}{s_{11}} \varepsilon_{11} - \left( \epsilon_{33} - \frac{d_{31}^2}{s_{11}} \right) \frac{\Delta\varphi_i}{a} \ . \tag{15}$$

Now find the average equation of dynamics for the medium

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{1}{a\,b} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \frac{\partial \hat{\sigma}_{12}}{\partial x_2} \, dx_2 dx_3 + \frac{1}{a\,b} \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \frac{\partial \hat{\sigma}_{13}}{\partial x_3} \, dx_2 dx_3 + \rho \omega^2 \hat{u}_1 = 0 \ . \tag{16}$$

Integrating it and taking into account the definitions of averages, we have:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{1}{a\,b} \int_{-a_1/2}^{a_1/2} \left( \sigma_{12}\left(x_2 = b_1/2\right) - \sigma_{12}\left(x_2 = -b_1/2\right) \right) dx_3$$

$$+ \frac{1}{a\,b} \int_{-b/2}^{b/2} \left( \hat{\sigma}_{13}\left(x_3 = a/2\right) - \hat{\sigma}_{13}\left(x_3 = -a/2\right) \right) dx_2 + \rho \omega^2 u_1 = 0 \ . \tag{17}$$

Since the stresses $\hat{\sigma}_{12}$ and $\hat{\sigma}_{13}$ on the corresponding side surfaces are equal to 0, both integrals become equal to 0 and the equation takes a typical form:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \rho \omega^2 u_1 = 0 \ . \tag{18}$$

Without changing the generality, it can be assumed that at the boundary $x_1 = 0$ and $x_1 = l$ some pre-known values of velocities are given, which are equal рівні $V_1$ and $V_2$ respectively, and directed in space as shown in Fig. 1. Then, after some almost trivial transformations, it can written that the displacements are equal to:

$$u_1 = - \frac{V_1 \, sin\left(\sqrt{\rho\,s_{11}}\,\omega\,(l-x_1)\right)}{j\omega\,sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \frac{V_2 \, sin\left(\sqrt{\rho\,s_{11}}\,\omega\,x_1\right)}{j\omega\,sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} \ . \tag{19}$$

In this case, the stress in the material is determined by the formula:

$$\sigma_{11} = \sqrt{\frac{\rho}{s_{11}}}\,\frac{V_1 \, cos\left(\sqrt{\rho\,s_{11}}\,\omega\,(l-x_1)\right)}{j\,sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \sqrt{\frac{\rho}{s_{11}}}\,\frac{V_2 \, cos\left(\sqrt{\rho\,s_{11}}\,\omega\,x_1\right)}{j\,sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \frac{d_{31}}{a\,b\,s_{11}} \sum_i b_i \, \Delta\varphi_i \ . \tag{20}$$

And the component of the dielectric displacement vector within the $i$-th electrode is equal to:

$$D_3^{(i)} = \sqrt{\frac{\rho}{s_{11}}}\,\frac{d_{31}\,V_1 \, cos\left(\sqrt{\rho\,s_{11}}\,\omega\,(l-x_1)\right)}{j\,sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \sqrt{\frac{\rho}{s_{11}}}\,\frac{d_{31}\,V_2 \, cos\left(\sqrt{\rho\,s_{11}}\,\omega\,x_1\right)}{j\,sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \left( \frac{d_{31}^2}{s_{11}} - \epsilon_{33} \right) \frac{\Delta\varphi_i}{a} \ . \tag{21}$$

In turn, the current flowing from the upper $i$-th electrode to the lower, will be found as the integral of the displacement vector

$$I_i = -j\omega \int_0^l \int_{S(b_i)} D_3^{(i)} dx_2 dx_1 = -\frac{d_{31} b_i\, V_1}{s_{11}} - \frac{d_{31} b_i\, V_2}{s_{11}} - j\omega\left(\frac{d_{31}^{\;2}}{s_{11}} - \epsilon_{33}\right)\frac{b_i\, l\, \Delta\varphi_i}{a}, \tag{22}$$

where $S(b_i)$ is the cross-section of the *i*-th electrode. A negative sign in the formula indicates the direction of current flow from the upper electrode to the lower, i.e., in the opposite direction relative to the direction of the displacement vector. Then, taking into account the definition of average stresses $\sigma_{11}$, the force generated by the piezoelectric element in cross section will be equal to

$$F(x_1) = a\,b\,\sigma_{11} = a\,b\sqrt{\frac{\rho}{s_{11}}}\frac{V_1\cos\left(\sqrt{\rho\,s_{11}}\,\omega\,(l-x_1)\right)}{j\,\sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + a\,b\sqrt{\frac{\rho}{s_{11}}}\frac{V_2\cos\left(\sqrt{\rho\,s_{11}}\,\omega\,x_1\right)}{j\,\sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \frac{d_{31}}{s_{11}}\sum_i b_i\,\Delta\varphi_i. \tag{23}$$

At the boundary $x_1 = 0$, the force generated by the piezoelectric element in cross section will be equal to:

$$F_1 = \sqrt{\frac{\rho}{s_{11}}}\frac{a\,b\,V_1\cos\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)}{j\,\sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \sqrt{\frac{\rho}{s_{11}}}\frac{a\,b\,V_2}{j\,\sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \frac{d_{31}}{s_{11}}\sum_i b_i\,\Delta\varphi_i. \tag{24}$$

And at the boundary $x_1 = l$, the force created by the piezoelectric element will be:

$$F_2 = \sqrt{\frac{\rho}{s_{11}}}\frac{a\,b\,V_1}{j\,\sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \sqrt{\frac{\rho}{s_{11}}}\frac{a\,b\,V_2\cos\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)}{j\,\sin\left(\sqrt{\rho\,s_{11}}\,\omega\,l\right)} + \frac{d_{31}}{s_{11}}\sum_i b_i\,\Delta\varphi_i. \tag{25}$$

We construct a substitution scheme that would model the last expressions taking into account the expression for currents through all electrodes (25). When constructing, we assume that the voltage corresponds to the force at the boundary of the element, and the velocity of the particles is modeled by the current. Suppose that the substitution scheme has the form shown in Fig. 2.

In this case, the transformers TVA and TVB have transformation coefficients equal to module 1, but their physical beginnings of the windings are joined counterclockwise, as shown in Fig. 2. Supports ZA and ZB simulate the relationships between particle velocities and forces arising at the boundaries of the element. TVC$_i$ transformers and ZC$_i$ supports simulate electrodes through which oscillations are excited. The corresponding circuits are connected in series in the same way as for the TVC$_1$ and TVC$_i$ transformers. The number of circuits coincides with the number of electrode pairs, and their parameters are determined by the characteristics of the electrodes. Thus, the transformation coefficient TVC$_i$ is equal to $2K_i$, where $K_i = \dfrac{d_{31} b_i}{s_{11}}$, and the complex resistance associated with the *i*-th electrode is determined as:

$$ZC_i = \cfrac{1}{j\omega\,\dfrac{b_i\, l}{a}\left(\epsilon_{33} - \dfrac{d_{31}^{\;2}}{s_{11}}\right)} = \cfrac{a\,s_{11}}{j\omega\,b_i\, l\left(\epsilon_{33} s_{11} - d_{31}^{\;2}\right)}. \tag{26}$$

Note that in the case where there is only one pair of electrodes on the surface, the value $b_1 = b$ is substituted instead of $b_i$ in the formula.
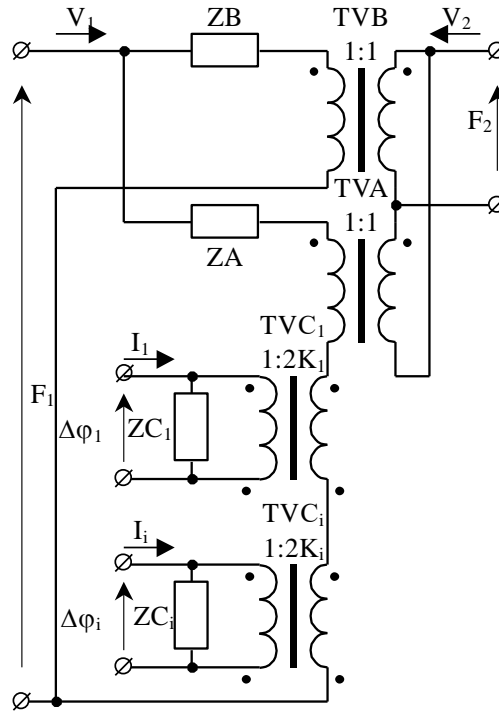
**Figure 2.** Scheme of substitution of one section with transverse polarization

It is easy to see that the circuit has the following relations between the model voltages $F_1$, $F_2$ and the model currents $V_1$, $V_2$.

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \dfrac{ZA + ZB}{4} & \dfrac{ZA - ZB}{4} \\ \dfrac{ZA - ZB}{4} & \dfrac{ZA + ZB}{4} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \sum_i K_i \, \Delta\varphi_i \,, \tag{27}$$

$$I_i = K_i (V_1 + V_2) + \frac{\Delta\varphi_i}{ZC_i} \,. \tag{28}$$

Hence the condition for resistance and the transformation coefficient

$$ZC_i = \frac{a}{j\omega \, b_i \, l} \frac{s_{11}}{\left(\epsilon_{33} s_{11} - d_{31}^{\,2}\right)}, \tag{29}$$

$$K = \frac{d_{31} b_i}{s_{11}} \tag{30}$$

$$ZA = \sqrt{\frac{\rho}{s_{11}}} \, \frac{2ab\left(cos\left(\sqrt{\rho\, s_{11}}\,\omega\, l\right) + 1\right)}{j\, sin\left(\sqrt{\rho\, s_{11}}\,\omega\, l\right)} = -j2ab\sqrt{\frac{\rho}{s_{11}}}\, ctg\left(\sqrt{\rho\, s_{11}}\,\omega\, l\,/\,2\right), \tag{31}$$

$$ZB = \sqrt{\frac{\rho}{s_{11}}} \, \frac{2\,a\,b\left(cos\left(\sqrt{\rho\, s_{11}}\,\omega\, l\right) - 1\right)}{j\, sin\left(\sqrt{\rho\, s_{11}}\,\omega\, l\right)} = j\,2\,ab\sqrt{\frac{\rho}{s_{11}}}\, tg\left(\sqrt{\rho\, s_{11}}\,\omega\, l\,/\,2\right). \tag{32}$$

Transcendental functions are not convenient to use. For example, they are difficult to represent in the form of an equivalent electrical circuit, because not all CAD are able to use arbitrary expressions to calculate frequency-dependent resistances, and so on. Therefore, we factorize the transcendental functions $ctg(x)$ and $tg(x)$ into series around special points, using the representations that can be found, for example, in [3], namely:

$$tg(x) = \sum_{k=1}^{\infty} \frac{8x}{\pi^2(2k-1)^2 - 4x^2} = \sum_{k=1}^{\infty} \frac{8x}{\pi^2(2k-1)^2\left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right)}, \tag{33}$$

$$ctg(x) = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2 - x^2} = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2x}{k^2\pi^2\left(1 - \frac{x^2}{k^2\pi^2}\right)}. \tag{34}$$

However, as shown by a numerical experiment, a simple break of a series on N terms gives a large error of the derivative near zero and thus distorts the approximation. Therefore, at the same time as discarding the members of a series with a number greater than N, we will add additional members to the finite amount:

$$\overline{tg(x,N)} = \sum_{k=1}^{N} \frac{8x}{\pi^2(2k-1)^2\left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right)} + A_t(N) + B_t(N)x + C_t(N)x^2, \tag{35}$$

where coefficients $A_t(N), B_t(N), C_t(N)$ will be determined in the way that discrepancy $g_t(x,N) = \overline{tg(x,N)} - tg(x)$ and its first derivatives were equal to 0 when $x \to 0$. Taking the boundary of the discrepancy and its first and second derivatives at $x \to 0$, and equating them to zero, we obtain a system of three linear equations, from which it follows that

$$A_t(N) = 0, \qquad B_t(N) = 1 - \sum_{k=1}^{N} \frac{8}{\pi^2(2k-1)^2}, \qquad C_t(N) = 0. \tag{36}$$

As a result, an expression of the approximation is obtained:

$$\overline{tg(x,N)} = \left(1 - \sum_{k=1}^{N} \frac{8}{\pi^2(2k-1)^2}\right)x + \sum_{k=1}^{N} \frac{8x}{\pi^2(2k-1)^2\left(1 - \frac{4x^2}{\pi^2(2k-1)^2}\right)}. \tag{37}$$

It is easy to see that with $N \to \infty$ the sum $\sum_{k=1}^{N} \frac{8}{\pi^2(2k-1)^2}$ approaches 1, turning the approximation into the initial one. Now consider the approximation $ctg(x)$. We discard the members of the initial series, leaving only the first N of them. And to increase the accuracy for small x, add an additional term to the finite amount:

$$\overline{ctg(x,N)} = \frac{1}{x} - \sum_{k=1}^{N} \frac{2x}{k^2 \pi^2 \left(1 - \frac{x^2}{k^2 \pi^2}\right)} + \frac{A_c(N)}{x} + B_c(N) + C_c(N)x + D_c(N)x^2 . \tag{38}$$

Since $ctg(x)$ goes to infinity when $x \to 0$, we choose the following expression as a function of discrepancy:

$$g_c(x,N) = x\left(\overline{ctg(x,N)} - ctg(x)\right) =$$
$$= 1 - \sum_{k=1}^{N} \frac{2x^2}{\left(\pi^2 k^2 - x^2\right)} + A_c(N) + B_c(N)x + C_c(N)x^2 + D_c(N)x^3 - x\,ctg(x). \tag{39}$$

The choice of such a representation is due to the fact that $\lim_{x \to 0}(ctg(x)x) = 1$. As for the previous function, we impose conditions that the discrepancy function and its derivatives from the first to the third order were equal to 0 when $x \to 0$. Herewith the system of linear equations is obtained, from which it follows that

$$A_c(N) = 0, \quad B_c(N) = 0, \quad C_c(N) = \sum_{k=1}^{N} \frac{2}{k^2 \pi^2} - \frac{1}{3}, \quad D_c(N) = 0. \tag{40}$$

Therefore, a more accurate approximation will have an expression

$$\overline{ctg(x,N)} = \frac{1}{x} - \left(\frac{1}{3} - \sum_{k=1}^{N} \frac{2}{k^2 \pi^2}\right)x - \sum_{k=1}^{N} \frac{2x}{k^2 \pi^2 \left(1 - \frac{x^2}{k^2 \pi^2}\right)} . \tag{41}$$

It is easy to verify that when with $N \to \infty$ the sum $\sum_{k=1}^{N} \frac{8}{\pi^2 (2k-1)^2}$ approaches s from below to $1/3$, converting approximation to the initial one within the boundary $N \to \infty$.

**Representation of circuit resistances for transverse polarization.** Using the obtained expansions of transcendental functions $\overline{tg(x,N)}$ and $\overline{ctg(x,N)}$ into series, we obtain:

$$ZA = \frac{4ab}{j\omega\, l\, s_{11}} + j\omega\, abl\, \rho \left(\frac{1}{3} - \sum_{k=1}^{N} \frac{2}{k^2 \pi^2}\right) + \sum_{k=1}^{N} \frac{j\omega\, 2\rho\, abl}{k^2 \pi^2 \left(1 - \frac{\rho\, s_{11} \omega^2\, l^2}{4k^2 \pi^2}\right)} . \tag{42}$$

Similarly,

$$ZB = \left(1 - \sum_{k=1}^{N} \frac{8}{\pi^2 (2k-1)^2}\right)abl\, \rho\, j\omega + \sum_{k=1}^{N} \frac{j8abl\rho\,\omega}{\pi^2 (2k-1)^2 \left(1 - \frac{\rho\, s_{11} \omega^2\, l^2}{\pi^2 (2k-1)^2}\right)} . \tag{43}$$
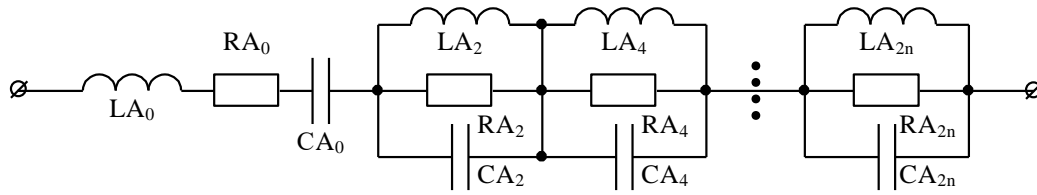
**Figure 3.** Representation of the substitution scheme ZA

We proceed to determining of representation of complex resistances ZA, ZB, ZC$_i$ in the form of substitution circuits. From the formula of complex resistance it follows that ZC$_i$ is a capacitance value

$$CC_i = \frac{b_i \, l \left( \epsilon_{33} s_{11} - d_{31}^{\,2} \right)}{a \, s_{11}}. \tag{44}$$

Now consider ZA. It is easy to see that the first term ZA is a capacitance value $CA_0 = \frac{s_{11} \, l}{4 \, a \, b}$. The second term $ZA_c$ can be represented as inductance

$$LA_0 = a \, b \, l \, \rho \left( \frac{1}{3} - \sum_{k=1}^{N} \frac{2}{k^2 \pi^2} \right). \tag{45}$$

To take into account the losses and the quality factor of the circuit Q, we introduce the loss resistance RA$_0$. As is known, the ohmic resistance of losses included in series in a series resonant circuit can be determined through the quality factor of the circuit as follows:

$$RA_0 = \frac{1}{Q} \sqrt{\frac{LA_0}{CA_0}} \,. = \frac{1}{Q} \sqrt{\frac{4 \, a^2 \, b^2 \, l \, \rho}{s_{11} \, l} \left( \frac{1}{3} - \sum_{k=1}^{N} \frac{2}{k^2 \pi^2} \right)} = \frac{2 \, a \, b}{Q} \sqrt{\frac{\rho}{s_{11}} \left( \frac{1}{3} - \sum_{k=1}^{N} \frac{2}{k^2 \pi^2} \right)}. \tag{46}$$

It is also easy to prove that the last ZA terms can be represented as series-connected parallel LC circuits. Indeed, the complex resistance of series-connected parallel LC circuits equals

$$\sum_{k=1}^{\infty} \frac{LA_k \, j\omega}{1 - LA_k \, CA_k \, \omega^2} = \sum_{k=1}^{\infty} \frac{j \, \omega \, 2 \, a \, b \rho \, l}{k^2 \pi^2 \left( 1 - \frac{\rho \, s_{11} \omega^2 l^2}{4k^2 \pi^2} \right)}. \tag{47}$$

Herewith it follows that

$$LA_k = \frac{2 \, a \, b\rho \, l}{k^2 \pi^2}, \quad CA_k = \frac{s_{11} \, l}{8 \, a \, b}. \tag{48}$$

To take into account the losses and the quality factor of the circle Q, we introduce the loss resistance RA$_k$. As it is known, the ohmic loss resistance connected in parallel to the parallel resonant circuit can be determined through the quality factor of the circuit as follows:

$$RA_k = Q\sqrt{\frac{LA_k}{CA_k}} = \frac{4\,a\,b\,Q}{k\,\pi}\sqrt{\frac{\rho}{s_{11}}}\,. \tag{49}$$

Thus, the substitution scheme ZA can be represented as shown in Fig. 4. In turn, ZB can also be represented as series-connected parallel LC circuits with which the inductor is connected in series. It is easy to determine that the value of the latter is equal

$$LB_0 = \left(1 - \sum_{k=1}^{N}\frac{8}{\pi^2(2k-1)^2}\right)a\,b\,l\,\rho\,. \tag{50}$$

Comparing the complex resistance of the parallel LC circuit and the components of the last sum ZB, we have:

$$LB_k = \frac{8\,a\,b\,l\,\rho}{(2k-1)^2\,\pi^2} \qquad CB_k = \frac{s_{11}\,l}{8\,a\,b}\,. \tag{51}$$

Completely similarly, entering the loss resistance in the circuit, we have

$$RB_k = Q\sqrt{\frac{LB_k}{CB_k}} = \frac{8\,a\,b\,Q}{(2k-1)\,\pi}\sqrt{\frac{\rho}{s_{11}}}\,. \tag{52}$$

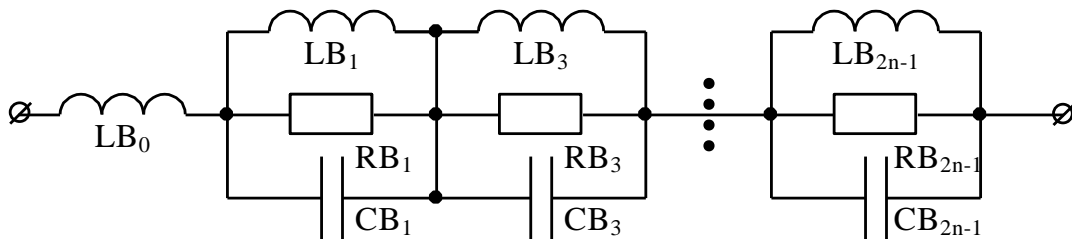Therefore, the substitution scheme ZB can be represented as it is shown in Fig. 4a.



**Figure 4.** Representation of the substitution scheme ZB

**General model of PT of transverse-transverse type.** We now construct a piezotransformer replacement circuit consisting of two sections with transverse polarization, provided that each section has only one electrode. At the boundary of the section, the force and velocity of the particles coincide in magnitude, so the corresponding contacts must be connected. On the free parts of the sections of the PT, forces are zero, which corresponds to a short circuit at the corresponding inputs.

Hereby, the general scheme of PT substitution can have the form presented in Fig. 6, a.
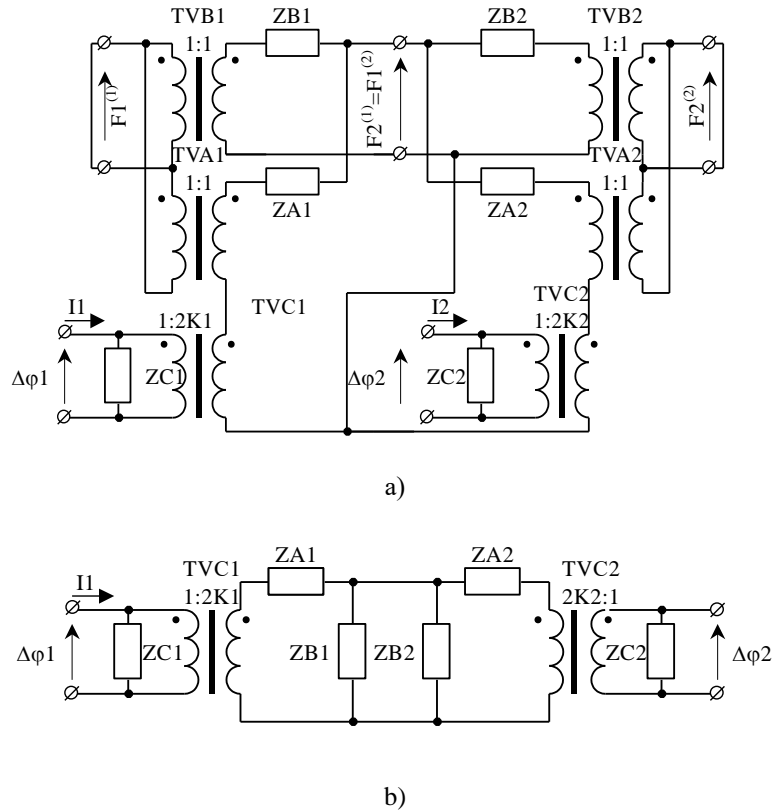
a)



b)

**Figure 5.** Scheme of replacement of piezotransformer of cross-transverse type

Short-circuited secondary windings of ideal transformers are equivalent to short-circuit of primary windings. Hence, the circuit will be significantly simplified, as the transformers TVA1 TVA2, TVB1, TVB2 become equivalent short circuits of the respective windings. Redrawing the scheme in a more convenient form, we obtain the classical scheme shown in Fig. 5, b. However, from the point of view of the use of CAD, it is much better to allow the user to combine individual sections of the PA, presented as separate elements. MicroCAP tools can be used to model PA sections. In this case, each individual section with its electrodes can be described by a macroelement, and then in the process of modeling the circuit to connect the terminals of the sections, which simulate mechanical oscillations with each other as simple electric circuits.

The scheme modeling a separate section of the piezoelectric element is shown in Fig. 6, and the use of sections, for example, to obtain frequency characteristics, is shown in Fig. 7.
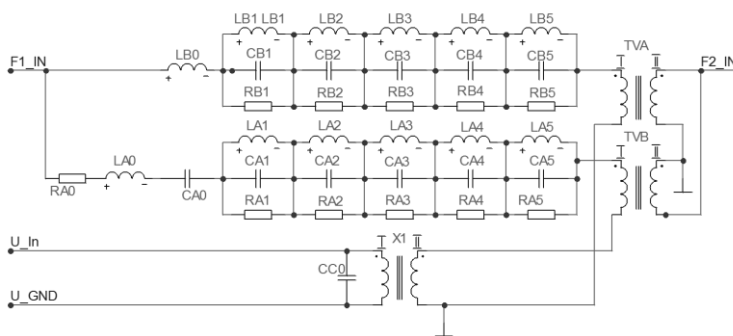


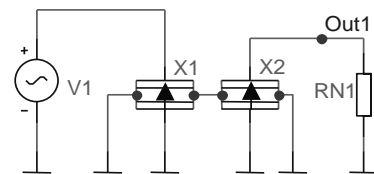**Figure 6.** The structure of the model of the piezoelectric section



**Figure 7.** Graphical representation of the connection of piezotransformer sections in MicroCap CAD

**Conclusions.** The proposed scheme and method of modeling, in contrast to the classical, given, for example, in [1, 2, 4], allows to take into account the presence of sections with different types of electrode placement, a large number of electrodes, and so on. The model allows to create a piezotransformer as a circuit consisting of standard blocks, which facilitates the design of the placement of electrodes and their coordination with the circuit.

The proposed model can easily take into account any number of resonances and the presence of several electrodes placed on the piezotransformer.

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# МОДЕЛЬ П'ЄЗОЕЛЕКТРИЧНИХ ТРАНСФОРМАТОРІВ ПОПЕРЕЧНО-ПОПЕРЕЧНОГО ТИПУ

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*Резюме. Представлено математичну модель п'єзоелектричного трансформатора поперечно-поперечного типу та описано методику її побудови. Хоча програми математичного моделювання п'єзоелектричних приладів можуть досягнути будь-якої наперед заданої точності моделювання, результати моделювання не можуть бути прямо використані у процесі розроблення радіоелектронної апаратури, тому що програми ніяк не інтегровані з САПР радіоелектронної апаратури. З цієї причини найчастіше при розрахунках та моделюванні схем на основі п'єзотрансформаторів використовують найпростішу еквівалентну схему, що адекватно відображає струми та напруги в колі п'єзотрансформатора лише в околі робочої резонансної частоти. В основу пропонованої моделі покладено одновимірне наближення рівнянь стану та динаміки п'єзоелектричного середовища для плоских пластин постійної товщини й ширини, що отримується з тривимірної системи рівнянь методом усереднення за шириною та товщиною. У той час, як звичайна наближена модель найчастіше дозволяє моделювати п'єзотрансформатор із двома парами електродів і лише в околі однієї резонансної частоти, побудована у статті модель дозволяє врахувати наявність кількох електродів на поверхнях п'єзотрансформатора та їх різне взаємне розміщення на верхній та нижній поверхнях п'єзопластини. У порівнянні зі звичайною, пропонована модель зручніша для проведення моделювання засобами систем схемотехнічного моделювання. Фактично, у розробленій моделі п'єзотрансформатор представлено як набір сполучених між собою секцій, котрі несуть на поверхнях одну чи кілька пар електродів. Також, на відміну від звичайної, пропонована модель дозволяє враховувати наявність кількох резонансних частот п'єзотрансформатора, що дозволяє адекватніше провести моделювання радіоелектронної апаратури, котра використовує у своїй структурі п'єзоелектричний трансформатор поперечно-поперечного типу. На основі математичної моделі побудовано схему заміщення окремих секцій п'єзоелектричного трансформатора та наведено формули для розрахунку параметрів елементів схеми. На прикладі показано реалізацію розробленої моделі в системі автоматизованого проектування MicroCAP.*

*Ключові слова: математична модель, п'єзотрансформатор, п'єзоелемент, MicroCAP.*