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# SEPARATION OF THE 3D STRESS STATE OF A LOADED PLATE INTO TWO-DIMENSIONAL TASKS: BENDING AND SYMMETRIC COMPRESSION OF THE PLATE

**Victor Revenko[1]; Andrian Revenko[2]**

*[1]The Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of the NAS of Ukraine, Lviv, Ukraine*
*[2]Karpenko Physico-Mechanical Institute of the NAS of Ukraine, Lviv, Ukraine*

***Summary***. *The three-dimensional stress-strain state of an isotropic plate loaded on all its surfaces is considered in the article. The initial problem is divided into two ones: symmetrical bending of the plate and a symmetrical compression of the plate, by specified loads. It is shown that the plane problem of the theory of elasticity is a special case of the second task. To solve the second task, the symmetry of normal stresses is used. Boundary conditions on plane surfaces are satisfied and harmonic conditions are obtained for some functions. Expressions of effort were found after integrating three-dimensional stresses that satisfy three equilibrium equations. For a thin plate, a closed system of equations was obtained to determine the harmonic functions. Displacements and stresses in the plate were expressed in two two-dimensional harmonic functions and a partial solution of the Laplace equation with the right-hand side, which is determined by the end loads. Three-dimensional boundary conditions were reduced to two-dimensional ones. The formula was found for experimental determination of the sum of normal stresses via the displacements of the surface of the plate.*

***Key words****: loaded plate, three-dimensional stressed state, stress tensor, Lamé equations.*

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**Statement of the problem.** The plates are widely used in the transport, energy engineering and construction industries. The development of science and technology puts forward new increased requirements for the accuracy of research on their strength and permissible deformations.

Therefore, there is a need for a fuller account of the nature of their load. Consequently, the original calculation models should be simplified by reducing the three-dimensional calculation of the plates to the study of their average two-dimensional stress state.

**Analysis of known results of three-dimensional studies of plates.** Plates, to which loads are applied both on the end planes and their sides, are widely used in building and engineering structures [1–3]. According to [4], when the end surface is loaded, the bending of thick plates should be considered as a three-dimensional problem of the theory of elasticity. If a thin plate is loaded only on its parallel sides and symmetrically to the middle surface, then its stress state is mainly calculated by the equations of the plane problem of the theory of elasticity [1, 2]. For thick plates, homogeneous solutions and the symbolic method are used [4], as well as the harmonic and biharmonic functions [5, 6], and the Papkovych – Neiber representation by the normal to the middle surface variable [7]. Reviews of the literature on models and methods for studying the stress state of plates have been given in [4, 5, 8]. In [9], the theory of a two-dimensional plane problem for thin and thick plates has been developed on the basis of the three-dimensional theory of elasticity, without using hypotheses about zero tangential stresses in the middle of the plate. In [10], the torque is taken into account and the theory of bending of a thick plate is developed on the basis of integration of a three-dimensional harmonic equation with an unknown right part, and in [11] stress is expressed through three two-dimensional functions satisfying equations in partial derivatives. Nowadays, the methods of reducing the three-dimensional stress state of the plate, under the action of normal loads given on all its surfaces, for the calculation of two-dimensional problems are not known.

**The purpose of research** is to express the general three-dimensional stress state of the plate, to the flat planes of which normal loads are applied, through two computational problems: bending and symmetrical end compression. For the second problem, a closed two-dimensional calculation model for calculating its stress state should be developed.

**Formulation of the problem and development of the initial system of equations.** A three-dimensional static problem of a loaded isotropic plate of constant thickness $h$ should be considered, its plane median surface occupies the area $S$ and coincides with the plane $Oxy$ of the Cartesian coordinate system: $x_1 = x,\ x_2 = y,\ x_3 = z$. Normal loads $q_j(x, y)$, $j = \overline{1.2}$ are applied to the plane end surfaces of the plate ($h_1 = h/2$, $h_2 = -h/2$), and tangents are absent. The side surface of the plate is denoted by $\Omega$. This initial problem should be divided into two ones. For the first, which describes the symmetrical bending of the plate, the normal loads of the plane surfaces of the plate are equal and directed in one direction:

$$\sigma_z(x, y, h_1) = g^+(x, y),\ \sigma_z(x, y, h_2) = -g^+(x, y), \tag{1}$$

and for the second problem, they are directed into the opposite direction

$$\sigma_z(x, y, h_j) = p^+(x, y), \quad \tau_{3m}(x, y, h_j) = 0,\ j, m = \overline{1.2} \tag{2}$$

where $g^+ = \frac{1}{2}(q_1 - q_2)$; $p^+ = \frac{1}{2}(q_1 + q_2)$; the signs «+», «–» describe the functions on the upper $z = h_1$ and lower $z = -h_1$ plane surfaces of the plate, respectively.

The reduction of the three-dimensional problem, which is described by relations (1), to the two-dimensional problem, is considered in [10, 11].

The load of the plate, which is determined by the boundary conditions (2) should be considered in detail. The loads (2) correspond to the normal stresses $\sigma_j$, $j = \overline{1.3}$, which are symmetrical with respect to the middle surface. Therefore, the following boundary conditions must be added to conditions (2) on the closed side surface $\Omega$ of the plate:

$$\sigma_n(x, y, z) = \sigma_{1n}|_\Omega,\ \tau_{nt}(x, y, z) = \sigma_{2n}|_\Omega,\ \tau_{nz}(x, y, z) = \sigma_{3n}|_\Omega, \tag{3}$$

where $n$ is normal to the surface $\Omega$, $\sigma_{jn}$, $j = \overline{1.3}$ are loads, $\sigma_{jn}(x, y, -z) = \sigma_{jn}(x, y, z)|_\Omega$, $i = \overline{1.2}$, $\sigma_{3n}(x, y, -z) = -\sigma_{3n}(x, y, z)|_\Omega$. The second problem, which is determined by the boundary conditions (2) – (3), will be considered the *symmetric compression of the plate on its ends*. If in conditions (2) put $p^+(x, y) = 0$, then we obtain a known plane problem in three-dimensional formulation is obtained, its reduction to the two-dimensional case is considered in

[9]. Thus, the plane problem of the theory of elasticity is a partial case of the symmetric compression of the plate on its ends.

To determine the problem of symmetric compression of a plate, a general representation of the solution of the Lame equations is used [9, 12]

$$u_x = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \quad u_y = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}, \quad u_z = \frac{\partial P}{\partial z} - 4(1-\nu)\Phi, \tag{4}$$

where $P = z\Phi + \Psi$; $\Phi$, $\Psi$, $Q$ are three-dimensional harmonic functions of displacements; $\nu$ is Poisson's ratio. Based on the representation of displacements (4), we obtain the expression of normal stresses

$$\sigma_j = 2G\left[\frac{\partial^2 P}{\partial x_j^2} - 2\nu\frac{\partial \Phi}{\partial x_3} - (-1)^j\frac{\partial^2 Q}{\partial x_1 \partial x_2}\right], \quad \sigma_3 = 2G\left[\frac{\partial^2 P}{\partial x_3^2} - 2(2-\nu)\frac{\partial \Phi}{\partial x_3}\right] \tag{5}$$

and tangential stresses

$$\tau_{12} = G\left[2\frac{\partial^2 P}{\partial x_1 \partial x_2} + \frac{\partial^2 Q}{\partial x_2^2} - \frac{\partial^2 Q}{\partial x_1^2}\right],$$

$$\tau_{j3} = G\left[\frac{\partial}{\partial x_j}\left[2\frac{\partial P}{\partial x_3} - 4(1-\nu)\Phi\right] - (-1)^j\frac{\partial^2 Q}{\partial x_{3-j}\partial x_3}\right], \quad j = \overline{1.2}. \tag{6}$$

The biharmonic function $P$ satisfies the following equation:

$$\Delta P + \frac{\partial^2}{\partial z^2}P = 2\frac{\partial}{\partial z}\Phi, \tag{7}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two-dimensional Laplace operator. For stresses representation (5) the following dependence is performed

$$\sigma_x + \sigma_y + \sigma_z = -2E\frac{\partial}{\partial z}\Phi. \tag{8}$$

Based on the relations (2) – (6), the functions $P$, $\Psi$, $Q$ will be even with respect to the variable $x_3$, and the function $\Phi$ will be odd ($\Phi^- = -\Phi^+$). After application of this symmetry and substituting stresses (5), (6) in the boundary conditions (2), the conditions on the upper surface of the plate are considered:

$$\frac{\partial^2 P^+}{\partial x_3^2} - 2(2-\nu)\frac{\partial \Phi^+}{\partial x_3} = \frac{1}{2G}p^+(x, y),$$

$$\frac{\partial}{\partial x_j}\left[2\frac{\partial P^+}{\partial x_3}-4(1-\nu)\Phi^+\right]-(-1)^j\frac{\partial^2 Q^+}{\partial x_{3-j}\partial x_3}=0\,,\quad j=\overline{1.2}\,. \tag{9}$$

From equations (9), the following conditions of two-dimensional harmony of values of functions on the top surface of the plate are followed:

$$\Delta[\frac{\partial P^+}{\partial z}-2(1-\nu)\Phi^+]=0\,,\quad \Delta\frac{\partial Q^+}{\partial z}=0\,. \tag{10}$$

The expressions of normal and tangential efforts are written in papers [2, 3, 9]. To do this, after integrating the stresses (5), (6) on thickness of the plate, the expressions of efforts are found that coincide with the corresponding expressions of efforts in the plane problem [9]:

$$T_x=2G\left[\frac{\partial^2\widetilde{P}}{\partial x^2}+\frac{\partial^2\widetilde{Q}}{\partial x\partial y}-4\nu\Phi^+\right],\;\; T_y=2G\left[\frac{\partial^2\widetilde{P}}{\partial y^2}-\frac{\partial^2\widetilde{Q}}{\partial x\partial y}-4\nu\Phi^+\right],$$

$$S_{xy}=S_{yx}=2G\left[\frac{\partial^2\widetilde{P}}{\partial x\partial y}+\frac{1}{2}(\frac{\partial^2\widetilde{Q}}{\partial y^2}-\frac{\partial^2\widetilde{Q}}{\partial x^2})\right], \tag{11}$$

where $\widetilde{P}=\int_{-h_1}^{h_1}Pdz$ , $\widetilde{Q}=\int_{-h_1}^{h_1}Qdz$ .

After using harmonic of the displacement functions (4) and integrating the equation (7) on the variable $z$ , the two-dimensional equations are deduced:

$$\Delta\widetilde{Q}=-2\frac{\partial Q^+}{\partial z}\,,\quad \Delta\widetilde{P}+2\frac{\partial P^+}{\partial z}=4\Phi^+\,. \tag{12}$$

After substituting the relationship (11) into the equilibrium equation of the plate in the efforts [2, 3] and after transformations, the key equations are obtained

$$\frac{\partial}{\partial x}[\Delta\widetilde{P}-4\nu\Phi^+]+\frac{1}{2}\Delta\frac{\partial\widetilde{Q}}{\partial y}=0\,,\quad \frac{\partial}{\partial y}[\Delta\widetilde{P}-4\nu\Phi^+]-\frac{1}{2}\Delta\frac{\partial\widetilde{Q}}{\partial x}=0\,. \tag{13}$$

From equations (13) we have obtained bigharmonic equations:

$$\Delta[\Delta\widetilde{P}-4\nu\Phi^+]=0\,,\quad \Delta\Delta\widetilde{Q}=0\,. \tag{14}$$

Based on relations (10), (12), equations (14) also have been derived:

In order to obtain a closed system of equations, it must be in mind that such inequalities will be performed for a thin plate

$$|T_x|,|T_y|>>\left|\int_{-h_1}^{h_1}\sigma_3(\mathrm{x,y,z})dz-hp^+(x,y)\right|. \tag{15}$$

After integrating the dependence (8) on the thickness of the plate and using relations (11), (15), the basic equation for the function $\tilde{P}$ is found

$$\Delta\tilde{P} = -4(1-\nu)\Phi^{+} - \frac{h}{2G}\mathrm{p}^{+}. \tag{16}$$

After substituting the dependence (16) into the first equation (14), the harmonic equation is deduced

$$\Delta[\Phi^{+} + \frac{h}{8G}p^{+}(\mathrm{x},\mathrm{y})] = 0. \tag{17}$$

A system of mutually agreed equations is developed: (9), (10), (12)–(14), (16), (17), the solutions of which shall be written in analytical form.

**Representation of stresses through two-dimensional harmonic functions.**

The general solution of the harmonic equation (17) is represented

$$\Phi^{+} = -\mathrm{h}\frac{\partial^2\varphi}{\partial y^2} - \frac{h}{8G}p^{+}(\mathrm{x},\mathrm{y}), \tag{18}$$

where $\varphi$ – an unknown two-dimensional harmonic function. Based on equations (10)

$$\frac{\partial P^{+}}{\partial z} = \mathrm{h}\frac{\partial^2\varphi_1}{\partial y^2} - (1-\nu)\frac{h}{4G}p^{+}, \tag{19}$$

where $\varphi_1$ – an unknown harmonic function. The relation (19) is applied and the last two equations (9) are simplified to this form

$$h\frac{\partial}{\partial x_j}\frac{\partial^2\varphi_2}{\partial y^2} = (-1)^j\frac{1}{2}\frac{\partial^2 Q^{+}}{\partial x_{3-j}\partial x_3}, \quad j = \overline{1.2}, \tag{20}$$

where $\varphi_2 = \varphi_1 + 2(1-\nu)\varphi$. After taking into account the harmonic of functions in relations (20), a simplification of their solutions are found

$$\frac{\partial Q^{+}}{\partial z} = -2h\frac{\partial^2\varphi_2}{\partial x\partial y}. \tag{21}$$

Based on the relation (21) and the first equation (12), a harmonic equation with the right part for determining the biharmonic function $\tilde{Q}$ is determined

$$\Delta\tilde{Q} = 4h\frac{\partial^2\varphi_2}{\partial x\partial y}. \tag{22}$$

After substituting the function (18) into equation (16), the second equation for determining the function $\tilde{P}$ on the basis of the harmonic equation with the right-hand side is determined

$$\Delta\tilde{P} = 4(1-\nu)\mathrm{h}\frac{\partial^2\varphi}{\partial y^2} - \frac{\nu h}{2G}p^+ . \tag{23}$$

After considering the dependences (14), (22) and the harmony of functions $\Delta\tilde{P} - 4\nu\Phi^+$, $\Delta\tilde{Q}$, and simplifying the relations(13), the formula is developed

We take into account dependencies (14), (22), the harmonic functions, simplified relations (13) and we get

$$[\Delta\tilde{P} - 4\nu\Phi^+] + 2h\frac{\partial^2\varphi_2}{\partial y^2} = 0 . \tag{24}$$

Based on expressions (18), (23) and equation (24), the relationship between the introduced harmonic functions are found

$$\varphi_1 = -2(2-\nu)\varphi ,\ \ \varphi_2 = -2\varphi . \tag{25}$$

After considering the dependence (25), the equation (22) is presented in a simpler form

$$\Delta\tilde{Q} = -8h\frac{\partial^2\varphi}{\partial x\partial y} . \tag{26}$$

The general representations of the solutions of equations (23), (26) are found

$$\tilde{P} = 2(1-\nu)hy\frac{\partial}{\partial y}\varphi + hg_1(x,y) + hf_1(x,y) ,\ \tilde{Q} = -4hy\frac{\partial\varphi}{\partial x} + hg_2(x,y) , \tag{27}$$

where the function $f_1$ is a partial solution of the equation

$$\Delta f_1 = -\frac{\nu}{2G}p^+ , \tag{28}$$

$g_j$, $j = \overline{1,2}$ are harmonic functions that can be represented as [9]

$$g_1 = (1+\nu)[\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x}],\quad g_2 = (1+\nu)[\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}] , \tag{29}$$

where $\phi$, $\psi$ are harmonic functions.

For both the symmetric compression of the plate on its ends and the plane problem, the expressions of normal and tangential forces (11) do not include the function $\phi$ of relations (29), so it can be ignored. Therefore, functions (27) can be written in the form

$$\tilde{P} = 2(1-\nu)hy\frac{\partial}{\partial y}\varphi - h(1+\nu)\frac{\partial\psi}{\partial x} + hf_1(x,y) ,\ \tilde{Q} = -4hy\frac{\partial\varphi}{\partial x} + h(1+\nu)\frac{\partial\psi}{\partial y} . \tag{30}$$

If you enter a biharmonic function according to the formula $U = 2E(y\varphi + \frac{\partial\psi}{\partial x})$, then from relations (11), (30) will be followed by known expressions for stresses of the plane problem of the theory of elasticity

$$\sigma_x = \frac{T_x}{h} = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{T_y}{h} = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = \frac{S_{xy}}{h} = -\frac{\partial^2 U}{\partial x \partial y}. \tag{31}$$

We substitute in relations (11) only expressions that depend on the function $p^+$, and we find the stresses that are determined by the end load

$$\sigma_x = -2G\frac{\partial^2 f_1}{\partial y^2}, \quad \sigma_y = -2G\frac{\partial^2 f_1}{\partial x^2}, \quad \tau_{xy} = 2G\frac{\partial^2 f_1}{\partial x \partial y}, \quad \sigma_x + \sigma_y = \nu p^+ \tag{32}$$

.

We are summing up stresses (31), (32) and obtain the general representation of the stresses for symmetric compression of the plate on its ends

$$\sigma_x = \frac{\partial^2}{\partial y^2}(U - 2Gf_1), \quad \sigma_y = \frac{\partial^2}{\partial x^2}(U - 2Gf_1),$$

$$\tau_{xy} = \tau_{yx} = -\frac{\partial^2}{\partial x \partial y}(U - 2Gf_1). \tag{33}$$

We write down the sum of normal stresses (33) and we get:

$$\sigma_x + \sigma_y = 4E\frac{\partial^2 \varphi}{\partial y^2} + \nu p^+ . \tag{34}$$

Conditions (9) on the plate surface are analytically satisfied during the development of the solution of the problem. The three-dimensional boundary conditions (3), are given on the side surface $\Omega$ of the plate, after integration in thickness are reduced to the conditions on the contour $L$ of the area $S$ [3, 9]. The stresses (33) are used and the boundary conditions on the contour $L$ of the plate are written according to [2]:

$$\{\sin^2\beta\sigma_y + \cos^2\beta\sigma_x + \sin 2\beta\tau_{xy}\}|_L = \sigma_n ,$$

$$\{\frac{\sin 2\varphi}{2}(\sigma_y - \sigma_x) + \cos 2\varphi\tau_{xy}\}|_L = \tau_n , \tag{35}$$

where $\sigma_n = \frac{1}{h}\int_{-h_1}^{h_1}\sigma_{1n}(x,y,z)dz\,|_L$, $\tau_n = \frac{1}{h}\int_{-h_1}^{h_1}\sigma_{2n}\tau_n(x,y,z)dz\,|_L$, $\beta$ is the angle between the normal to the contour $L$ and the axis $Ox$ .

Since all the relations of the theory of elasticity are precisely satisfied, the displacements in the plate is found in the analytical form after averaging formulas (4) and using relations (19), (30). The displacement of the upper plane surface of the plate will have an expression

$$u_z^+ = -2\nu hE\frac{\partial^2\varphi}{\partial y^2} + (1-\nu)\frac{h}{4G}p^+ . \quad (36)$$

Based on the formula (34), the displacement $u_z^+$ is written in a different form

$$u_z^+ = \frac{h}{2E}[p^+ - \nu(\sigma_x + \sigma_y)] . \quad (37)$$

For the plane problem ( $p^+ = 0$ ), formula (37) will be more accurate

$$u_z^+ = -\frac{h\nu}{2E}(\sigma_x + \sigma_y) . \quad (38)$$

To experimentally determine the sum of normal stresses in the plate, the formula (38) can be used. Formula (38) allows to experimentally determine the stress concentration on the load-free ( $\sigma_n = 0$ ) contour of the hole in the plate. We take into account that on the load-free contour of the hole, equality is performed

$$\sigma_\tau = \sigma_x + \sigma_y , \quad (39)$$

where $\sigma_\tau$ is the unknown stress along the contour of the hole, directed perpendicular to the normal to the contour. Then, based on equations (38), (39) the stresse is determined

$$\sigma_\tau = -\frac{2E}{h\nu}u_z^+ . \quad (40)$$

Based on formula (40), after experimental measurement of the displacement $u_z^+$ along the contour of the hole, the value $\sigma_\tau$ could be found.

The partial solution of equation (28) is assumed to be known, a boundary value problem (35) for determining harmonic functions $\varphi$, $\psi$ is solved, and the averaged stresses on the middle surface of the plate are found. Based on these stresses, the deformations and displacements of the plate surfaces are determined.

**Conclusions**. For the first time, within the framework of three-dimensional theory of elasticity on the basis of the general solution of Lame's equations, without using the hypotheses on the distribution of displacements and stresses, the two-dimensional theory of the symmetric compression of a plate at its ends has been determined. The efforts have been expressed through two biharmonic functions with known right parts. The theory of compression of plates is proposed, which explicit takes into account the ends loads. The formula has been found for the normal displacements of the plate surface, based on it the sum of normal stresses in the plate could be experimentally determined. The obtained results can be used to calculate the stress state of the plates.

**References**

1. Sadd M. H. Elasticity. Theory, applications, and numerics, Amsterdam: Academic Press: 2014. 600 p.
2. Donell L. H. Beams, plates and shells. New York: McGraw-Hill: 1976. 568 p.
3. Timoshenko S. P., Woinowsky-Krieger S. Theory of Plates and Shells, second edition, New York: McGraw-Hill, 1959.595 p.
4. Shaldyrvan V. A. Some results and problems in the three-dimensional theory of plates (Review), Int. Appl. Mech. Vol. 43. No. 2. 2007. P. 160–181. DOI: https://doi.org/10.1007/s10778-007-0014-x
5. Meleshko V. V. Selected topics in the history of the twodimensional biharmonic problem. Applied Mechanics Reviews. 56 (1). 2003. P. 33–85. DOI: https://doi.org/10.1115/1.1521166
6. Grinchenko V. T. The biharmonic problem and progress in the development of analytical methods for the solution of boundary-value problems, The Journal of Engineering Mathematics, 46, No. 3. 2003. P. 281–297. DOI: https://doi.org/10.1023/A:1025049526317
7. Wang W., Shi M. X. Thick plate theory based on general solutions of elasticity, Acta Mechanica. 123. 1997. P. 27–36. DOI: https://doi.org/10.1007/BF01178398
8. Kobayashi H. A. Survey of Books and Monographs on Plates, Mem. Fac. Eng., Osaka City Univ. 38. 1997. P. 73–98.
9. Revenko V. P., Revenko A. V. Determination of Plane Stress-Strain States of the Plates on the Basis of the Three-Dimensional Theory of Elasticity, Materials Science. Vol. 52. No. 6. 2017. P. 811–818. DOI: https://doi.org/10.1007/s11003-017-0025-7
10. Revenko V. P. Reduction of a three-dimensional problem of the theory of bending of thick plates to the solution of two two-dimensional problems. Materials Science. 51. No. 6. 2015. P. 785–792. DOI: https://doi.org/10.1007/s11003-016-9903-7
11. Revenko V. P. Development of two-dimensional theory of thick plates bending on the basis of general solution of Lamé equations, Scientific journal of the Ternopil National Technical University. No. 1 (89). 2018. P. 33–39. DOI: https://doi.org/10.33108/visnyk_tntu2018.01.033
12. Revenko V. P. Solving the three-dimensional equations of the linear theory of elasticity. Int. Appl. Mech. Vol. 45. No. 7. 2009. P. 730–741. DOI: https://doi.org/10.1007/s10778-009-0225-4

**Список використаної літератури**

1. Sadd M. H. Elasticity. Theory, applications, and numerics, Amsterdam: Academic Press: 2014. 600 p.
2. Donell L. H. Beams, plates and shells. New York: McGraw-Hill: 1976. 568 p.
3. Timoshenko S. P., Woinowsky-Krieger S. Theory of Plates and Shells, second edition, New York: McGraw-Hill, 1959.595 p.
4. Shaldyrvan V. A. Some results and problems in the three-dimensional theory of plates (Review), Int. Appl. Mech. Vol. 43. No. 2. 2007. P. 160–181. DOI: https://doi.org/10.1007/s10778-007-0014-x
5. Meleshko V. V. Selected topics in the history of the twodimensional biharmonic problem. Applied Mechanics Reviews. 56 (1). 2003. P. 33–85. DOI: https://doi.org/10.1115/1.1521166
6. Grinchenko V. T. The biharmonic problem and progress in the development of analytical methods for the solution of boundary-value problems, The Journal of Engineering Mathematics, 46, No. 3. 2003. P. 281–297. DOI: https://doi.org/10.1023/A:1025049526317
7. Wang W., Shi M. X. Thick plate theory based on general solutions of elasticity, Acta Mechanica. 123. 1997. P. 27–36. DOI: https://doi.org/10.1007/BF01178398
8. Kobayashi H. A. Survey of Books and Monographs on Plates, Mem. Fac. Eng., Osaka City Univ. 38. 1997. P. 73–98.
9. Revenko V. P., Revenko A. V. Determination of Plane Stress-Strain States of the Plates on the Basis of the Three-Dimensional Theory of Elasticity, Materials Science. Vol. 52. No. 6. 2017. P. 811–818. DOI: https://doi.org/10.1007/s11003-017-0025-7
10. Revenko V. P. Reduction of a three-dimensional problem of the theory of bending of thick plates to the solution of two two-dimensional problems. Materials Science. 51. No. 6. 2015. P. 785–792. DOI: https://doi.org/10.1007/s11003-016-9903-7
11. Revenko V. P. Development of two-dimensional theory of thick plates bending on the basis of general solution of Lamé equations, Scientific journal of the Ternopil National Technical University. No. 1 (89). 2018. P. 33–39. DOI: https://doi.org/10.33108/visnyk_tntu2018.01.033
12. Revenko V. P. Solving the three-dimensional equations of the linear theory of elasticity. Int. Appl. Mech. Vol. 45. No. 7. 2009. P. 730–741. DOI: https://doi.org/10.1007/s10778-009-0225-4

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## РОЗДІЛЕННЯ ТРИВИМІРНОГО НАПРУЖЕНОГО СТАНУ НАВАНТАЖЕНОЇ ПЛАСТИНИ НА ДВОВИМІРНІ ЗАДАЧІ: ЗГИНУ І СИМЕТРИЧНОГО СТИСКУ ПЛАСТИНИ

**Віктор Ревенко[1]; Андріан Ревенко[2]**

*[1]Інститут прикладних проблем механіки і математики імені Я. С. Підстригача НАН України, Львів, Україна*
*[2]Фізико-механічний інститут імені Г. В. Карпенка НАН України, Львів, Україна*

***Резюме.*** *Вивчено тривимірний напружено-деформований стан ізотропної пластини, навантаженої на всіх своїх поверхнях. Розглянуто випадок, коли до її плоских торцевих поверхонь прикладені нормальні навантаження, а дотичні – відсутні. Для описування напруженого стану використано лінійну математичну модель тривимірного ізотропного тіла за відсутності об'ємних сил. Розглянута модель деформованого тіла базується на поданні переміщень і напружень через три гармонічні функції переміщень, які описують загальний розв'язок рівнянь Ляме. Вихідну задачу поділено на дві: симетричний згин пластини і симетричний торцевий стиск пластини заданими навантаженнями. Показано, що частковим випадком другої задачі є плоска задача теорії пружності. Для розв'язання другої задачі використано симетричність нормальних напружень відносно серединної поверхні пластини. Враховано цю симетрію і задоволення крайових умов на плоских поверхнях зведено до трьох умов, з яких отримано умови двовимірної гармонічності значень введених функцій на верхній плоскій поверхні пластини. Після інтегрування тривимірних напружень уздовж нормалі до серединної поверхні пластини знайдено вирази зусиль. Підставлено зусилля в рівняння рівноваги пластини і після перетворень отримано два бігармонічних рівняння. Показано, що для тонкої пластини будуть наближено виконуватися відповідні нерівності на нормальні напруження, після використання яких отримано замкнену систему рівнянь. Встановлено, що отримані рівняння узгоджені між собою. Записані співвідношення виражені через двовимірні гармонічні функції. Записано крайову задачу для визначення гармонічних функцій. Переміщення й напруження в пластині виражено через дві двовимірні гармонічні функції й частковий розв'язок рівняння Лапласа з відомою функцією, яка визначається торцевими навантаженнями. Аналогічно як в плоскій задачі, напруження виражені через функцію напружень і частковий розв'язок. Тривимірні крайові умови зведені до двовимірного вигляду. Знайдено формулу для експериментального визначення суми нормальних напружень у пластин через експериментально виміряні переміщення бічної поверхні пластини.*

***Ключові слова:*** *навантажена пластина, тривимірний напружений стан, тензор напружень, рівняння Ляме.*

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