Вісник Тернопільського національного технічного університету
https://doi.org/10.33108/visnyk_tntu
Scientific Journal of the Ternopil National Technical University
2024, № 1 (113) https://doi.org/10.33108/visnyk_tntu2024.01
ISSN 2522-4433. Web: ***visnyk.tntu.edu.ua***

UDC 517.51

PERIODIC FUNCTIONS WITH VARIABLE PERIOD – BASIC CONCEPTS AND CERTAIN INVESTIGATION RESULTS

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Summary. *Investigation of real signals is one of the most important applied areas of mathematics. According to their properties, signals are quite diverse, and methods of their research are different as well. Among this diversity, periodic signals with variable period make up a significant proportion. Till present, no attention was paid to the theory of such signals. In this paper, we define periodic functions with variable period, which are the model of these signals. Some properties of the variable period are considered. Examples of the analytical formulation of functions with variable period in the form of trigonometric functions with variable period are given and their variable periods are recorded. It is pointed out that these functions can be used as basic ones for constructing orthogonal system of trigonometric functions with variable period and its use for constructing Fourier series of functions with variable period.*

Key words: *periodic functions with variable period, variable period, trigonometric functions with variable period.*

https://doi.org/10.33108/visnyk_tntu2024.01.046 *Received 03.01.2024*

Statement of the problem. One of the most important applied areas of mathematics is the investigation of real signals. By their nature, the properties of signals are quite diverse, accordingly, the methods of their research are also different. Among this diversity, **periodic** signals, the **period** of which is considered to be **constant**, make up a significant proportion. The model of periodic signals is the periodic functions for which their theory, known as the theory of Fourier series, is constructed and used to investigate real periodic signals.

However, in addition to signals with a constant period, signals that are somewhat similar to periodic ones are encountered in applied research. The signal values are repeated, but the period of repetition is no longer constant, but changes in a certain way. We will call such signals **periodic signals with variable period**. A vivid example of such signals are the well-known electrocardiograms obtained during or after the patient's body is exposed to physical activity (or other «resting» stimulus). Ambulance sirens and air alarm sirens also have variable period.

How to investigate the variable period signals? The review of the literature sources shows that, **no attention was paid** to the issues of their theory.

It is shown by practice that in order to obtain satisfactory results in this area, it is reasonable to use time-tested approach, the essence of which is reduced to the **«model-algorithm-program»** triad. According to this approach, **at the first** stage of the triad, the signal model is substantiated (in our case, the model of periodic signals with variable period), at **the second** stage, analytical methods and algorithms for its investigation (analysis) are developed, and at **the third** stage, the corresponding software is created.

Analysis of the available results. The first significant steps in the investigation of signals with variable periods have been made. Thus, in paper [1] for the first time, the class of periodic functions with variable period, which can be used as the model of corresponding signals was introduced. Examples of trigonometric functions with variable period are given in paper [2], and orthogonal system of trigonometric functions witha variable period is written on this basis of [3, 4]. Later, a number of other results of the theory of periodic functions with variable period and their application to the solution of applied problems were obtained.

The objective of the paper is to review the main results of the theory of periodic functions with variable period and to outline the ways of further research.

Presentation of the material. The fundamental step in the investigation of periodic signals with variable period is the substantiation of their model. Such model was introduced in paper [1] in the form of periodic functions with variable period.

Definition 1. Function $f(x), x \in I \subseteq R$, is called **periodic function with variable period,** if there is differentiable function $T(x) > 0$ such that for all $x \in I$ such that $x + T(x) \in I$, the equality is

$$f(x) = f(x + T(x)). \tag{1}$$

Function $T(x)$ is called variable period.

It follows from (1) that at $T(x) = T = const$ function f is periodic with period T in the usual sense.

Example of variable period $T(x)$ graph is shown in Figure 1. At point x_1, the period of function $f(x)$ is $T(x_1)$, so the values of the function at points x_1 and $x_1 + T(x_1)$ are equal to $f(x_1) = f(x_1 + T(x_1))$. At point x_2, the period is the number $T(x_2)$. The figure shows that the periods at points x_1 and x_2 are different. For comparison, the constant period $T = const$ is also shown in Figure.

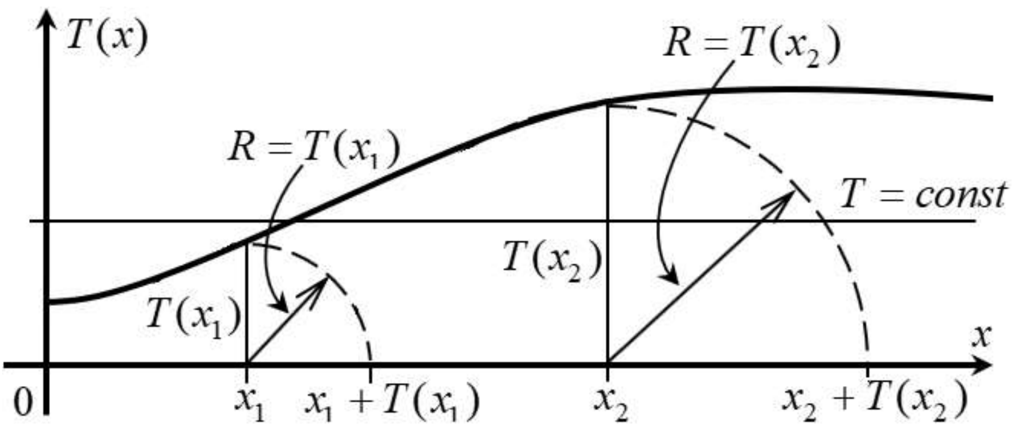

Figure 1. Variable period $T(x)$ its values at points x_1 and x_2 and the corresponding points $x_1 + T(x_1)$ and $x_2 + T(x_2)$, where the function values are repeated. For comparison, the constant period $T = const$

Variable period $T^{-}(x)$. Let is note that for periodic function $g(x)$ with period T, the equality is

$$g(x) = g(x + T) = g(x - T).$$

For function $f(x)$ with variable period $T(x)$, the same equality $f(x)=f(x+T(x))=f(x-T(x))$ does not hold in general. Therefore, in order to find the point located one period to the left of x, where the value of the function is repeated, it is necessary to introduce another variable period $T^{-}(x)$, such that we must introduce another variable period, such as

$$f(x)=f\left(x-T^{-}(x)\right).$$

You can make sure that the ratio is fulfiled for variable periods $T(x)$ and $T^{-}(x)$:

$$T(x)=T^{-}(x+T(x)),$$
$$T^{-}(x)=T\left(x-T^{-}(x)\right).$$

Properties of the variable period $T(x)$. Function $T(x)$, as variable period of function $f(x)$, should satisfy certain conditions. As mentioned in the definition, it must be continuous. It is also easy to show that its derivative must be greater than -1: $T'(x)>-1$. Indeed, since $x+\Delta x>x$, then for the corresponding points $x+T(x)$ and $x+\Delta x+T(x+\Delta x)$, taken through the period, the inequality $x+\Delta x+T(x+\Delta x)>x+T(x)$ or $T(x+\Delta x)-T(x)>-\Delta x$ should also be fulfilled. It follows from the last inequality that $\frac{T(x+\Delta x)-T(x)}{\Delta x}>-1$. At the limit transition, we get that for variable period $T(x)$ is its derivative the inequality or must also hold for the corresponding points and, taken at intervals. The last inequality implies that. At the limit transition, we obtain that for a variable period its derivative

$$T'(x)>-1,\ x\in I\,. \tag{2}$$

It can be seen from (2) that when period $T(x)$ decreases at certain intervals, this decrease should be slower than the decrease of function $y(x)=-x$. In some cases, the properties of the variable period $T(x)$ can be specified.

Analytical definition of periodic functions with variable period. The simplest periodic functions with variable period that can be defined analytically are trigonometric functions

$$\sin\, x^{\alpha},\ \cos x^{\alpha},\ tgx^{\alpha},\ ctgx^{\alpha},\ x\in I=[0,\infty),\ \alpha>0\,.$$

At $\alpha=1$ we get the usual trigonometric functions. The restriction on argument x is caused by the fact that at $x<0$, depending on whether parameter α, is even or odd, the domain of the definition I is specified. For even α, the domain is $I=[0,\infty)$, for odd α, the domain is $I=(-\infty,\infty)$. In order not to discuss this every time, we assume that the domain is $I=[0,\infty)$.

Example 1: Function $f_1(x) = \sin x^{3/4}$ (thick line) and for comparison function $f_2(x) = \sin x$ (thin line). Are shown in Figure 2.

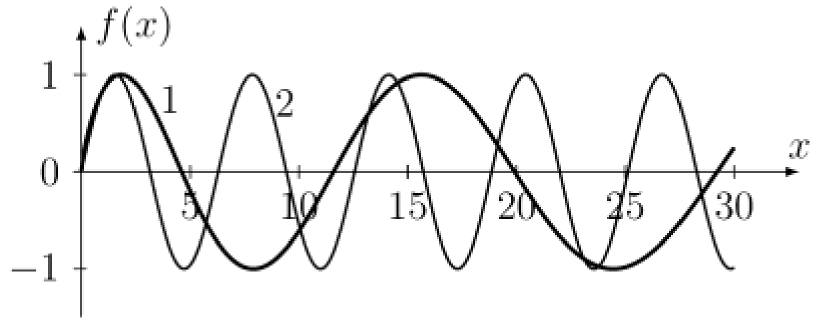

Figure 2. Function $f_1(x) = \sin x^{3/4}$ (thick line), $f_2(x) = \sin x$ (thin line)

Analyzing the graphs, it can be seen that function $f(x) = \sin x^{3/4}$ «stretches» as the argument increases, that is, its period increases. **Two** periodic oscillations fit into the interval $[0,30]$ for this function, and **more than four** oscillations fit into the same interval for function $\sin x$.

Example 2: For the case when $\alpha > 1$, Figure 3 shows function $f_1(x) = \sin x^{4/3}$ (thick line) and function $f_2(x) = \sin x$ (thin line). It is evident from the figure that as the argument increases, the function «compresses», that is, the period decreases. If sinusoid $f_2(x) = \sin x$ on interval $[0,15]$ makes more than **two** periodic oscillations, then $f_1(x) = \sin x^{4/3}$ includes **more than five and a half** oscillations.

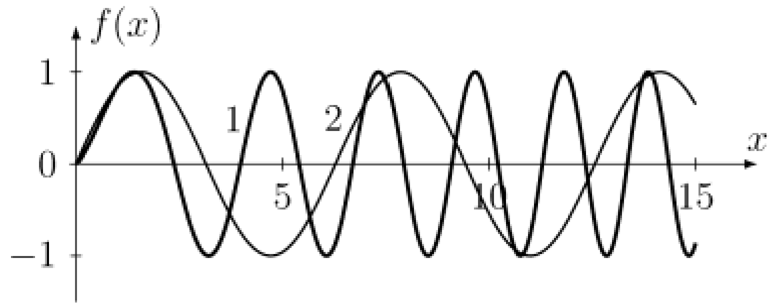

Figure 3. Graphs of function: $f_1(x) = \sin x^{4/3}$ (thick line); $f_2(x) = \sin x$ (thin line)

Variable periods of sinusoidal functions. Considering examples of periodic functions with variable period $f_1(x) = \sin x^{3/4}$ and $f_1(x) = \sin x^{4/3}$, visual analysis of the patterns of changes in their periods was carried out. Let us consider the problem of what are the analytical expressions of the periods of this kind of functions. Let us show what happens

Lemma. Variable periods of functions $\sin x^{\alpha}$ and $\cos x^{\alpha}$, $x \geq 0, \alpha > 0$, are defined by the following formulas:

$$T(x) = -x + \left(x^{\alpha} + 2\pi\right)^{1/\alpha},\ x \in [0, \infty), \tag{3 a}$$

$$T^{-}(x) = x - \left(x^{\alpha} - 2\pi\right)^{1/\alpha},\ x \in [T(0), \infty) = \left[(2\pi)^{1/\alpha}, \infty\right). \tag{3 b}$$

Proof. To prove (3a), we write down the condition for the periodicity of function $\sin x^{\alpha}$, $x \geq 0$, taking into account that the period of sinusoidal functions is 2π :

$$\sin\left(x + T(x)\right)^{\alpha} = \sin\left(x^{\alpha} + 2\pi\right).$$

Equating the arguments of this equality, we get

$$\left(x + T(x)\right)^{\alpha} = x^{\alpha} + 2\pi .$$

If the right and left sides of this equality are raised to power $1/\alpha$, we have $x + T(x) = \left(x^{\alpha} + 2\pi\right)^{1/\alpha}$. Hence, we obtain expression (3a):

$$T(x) = -x + \left(x^{\alpha} + 2\pi\right)^{1/\alpha},\ x \in [0, \infty). \tag{3 a}$$

Let us note that at point $x = 0$ the period is

$$T(0) = (2\pi)^{1/\alpha} . \tag{4}$$

Similarly, we obtain formula (3b) of variable period $T^{-}(x)$. To do this, let's write down the condition for the periodicity of function $\sin x^{\alpha}$ for the case when its argument decreases:

$$\sin\left(x - T^{-}(x)\right)^{\alpha} = \sin\left(x^{\alpha} - 2\pi\right).$$

The arguments for this equality should coincide, i.e.

$$\left(x - T^{-}(x)\right)^{\alpha} = x^{\alpha} - 2\pi .$$

Raising the right and left parts of this expression to power $1/\alpha$, we get $x - T^{-}(x) = \left(x^{\alpha} - 2\pi\right)^{1/\alpha}$. From the last equality, we have that the period is

$$T^{-}(x) = x - \left(x^{\alpha} - 2\pi\right)^{1/\alpha},\ x \in [T(0), \infty) = \left[(2\pi)^{1/\alpha}, \infty\right). \tag{3 b}$$

About the interval of determination of the period $T^{-}(x)$. Let us consider why, unlike interval $I=[0,\infty)$ where functions $\sin x^{\alpha}$, $\cos x^{\alpha}$ and period $T(x)$ are defined, for period $T^{-}(x)$ its determination domain is interval $I=[T(0),\infty)=\left[(2\pi)^{1/\alpha},\infty\right)$.

If we consider the condition of periodicity for function $\sin x^{\alpha}$, $\cos x^{\alpha}$, $x\geq 0$, as the argument decreases, i.e., move along axis Ox from right to left in steps equal to period $T^{-}(x)$, then we can get to the left boundary point of the interval $I=[0,\infty)$, i.e., to point $x=0$, from point $x=T(0)=(2\pi)^{1/\alpha}$. Indeed, taking into account (4b) and equality (4), at point $x=T(0)$ period is

$$T^{-}(T(0))=T(0)-\left(T(0)^{\alpha}-2\pi\right)^{1/\alpha}=T(0)-(2\pi-2\pi)^{1/\alpha}=T(0).$$

If we move with the same length steps $T^{-}(x)$ from all other points of interval $[0,T(0))$, we will go outside interval $I=[0,\infty)$, that is, we will leave the area of definition of functions $\sin x^{\alpha}$, $\cos x^{\alpha}$. From the above mentioned considerations, it turns out that for period $T^{-}(x)$, its area of definition is interval $I=[T(0),\infty)$, which is reflected in formula (4 b).

Verification of lemma statements. The statement that for functions $\sin x^{\alpha}$ and $\cos x^{\alpha}$ their periods are expressed by formulas (4a) and (4b) is easily verified. For example, for function $\sin x^{\alpha}$ we have

$$\sin\left(x+T(x)\right)^{\alpha}=\sin\left(\left(x-x+\left(x^{\alpha}+2\pi\right)\right)^{1/\alpha}\right)^{\alpha}=$$

$$=\sin\left(\left(x^{\alpha}+2\pi\right)^{1/\alpha}\right)^{\alpha}=\sin\left(x^{\alpha}+2\pi\right)=\sin x^{\alpha}.$$

The same function $\sin x^{\alpha}$ is repeated after period $T^{-}(x)$:

$$\sin\left(x-T^{-}(x)\right)^{\alpha}=\sin\left(\left(x-x+\left(x^{\alpha}-2\pi\right)\right)^{1/\alpha}\right)^{\alpha}=$$

$$=\sin\left(\left(x^{\alpha}-2\pi\right)^{1/\alpha}\right)^{\alpha}=\sin\left(x^{\alpha}-2\pi\right)=\sin x^{\alpha}.$$

Similar equalities are checked for function $\cos x^{\alpha}$.

For functions tgx^{α}, $ctgx^{\alpha}$, variable periods are expressed by formulas:

$$T(x)=-x+\left(x^{\alpha}+\pi\right)^{1/\alpha},\ x\in[0,\infty), \tag{5 a}$$

$$T^{-}(x)=x-\left(x^{\alpha}-\pi\right)^{1/\alpha},\ x\in[T(0),\infty)=\left[\pi^{1/\alpha},\infty\right). \tag{5 b}$$

Examples of variable periods of sinusoidal functions.

Example 3. Let's write the variable periods for the functions given in examples 1 and 2. Based on (3a) for function $\sin x^{3/4}$, their variable period is

$$T_{3/4}(x) = -x + \left(x^{3/4} + 2\pi\right)^{4/3}, \; x \in [0, \infty).$$

Taking into account (3b) and value $T(0) = (2\pi)^{4/3} \approx 11.594$, the variable period is

$$T^{-}_{3/4}(x) = x - \left(x^{3/4} - 2\pi\right)^{4/3}, \quad x \ge (2\pi)^{4/3} \approx 11.594 .$$

Graphs of these periods are shown in Figure 4. For comparison, period $T = 2\pi$ of function $\sin x$ is also given. It can be seen from the figure that period $T(x)$ increases with the increase of the argument, which confirms the previously expressed considerations while analyzing the graphs in Figure 2. The calculation of the period values for specific points also shows its growth. So at point $x = 0$ of the period value $T(0) \approx 11.594$, for $x = 30$ the period is much larger $T(30) \approx 21.062$. Period $T^{-}(x)$ decreases with the argument decreases

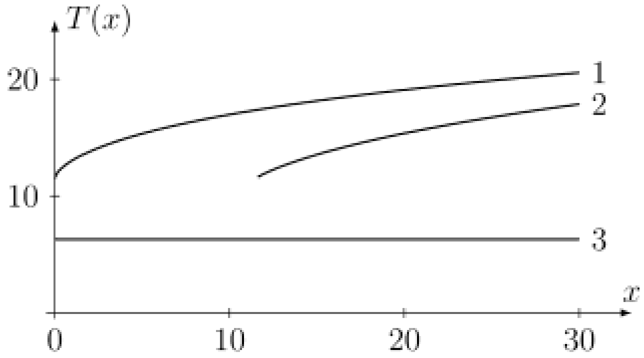

Figure 4. Variable periods for function $\sin x^{3/4}$: $T_{3/4}(x), \; x \ge 0$ (graph 1); $T^{-}_{3/4}(x), \; x \ge (2\pi)^{4/3} \approx 11.594$ (graph 2). For comparison period $T = 2\pi$ for function $\sin x$ (graph 3)

Example 4. For function $f(x) = \sin x^{4/3}, \; x \ge 0$, its variable periods are determined by formulas

$$T_{4/3}(x) = -x + \left(x^{4/3} + 2\pi\right)^{3/4}, \; x \ge 0,$$

$$T^{-}_{4/3}(x) = x - \left(x^{4/3} - 2\pi\right)^{3/4}, \quad x \ge T_{4/3}(0) = (2\pi)^{3/4} \approx 3.968 .$$

The graphs of the periods are shown in Figure 5. Unlike the previous case, here period $T_{4/3}(x)$ decreases with the argument growth. If, for example, for $x = 0$ period is $T_{4/3}(0) = 3.968$, then at point $x = 15$ period is $T_{4/3}(15) = 1.873$. On the contrary, period $T^{-}_{4/3}(x),\ x \geq (2\pi)^{3/4} \approx 3.968$ increases as the argument **decreases**.

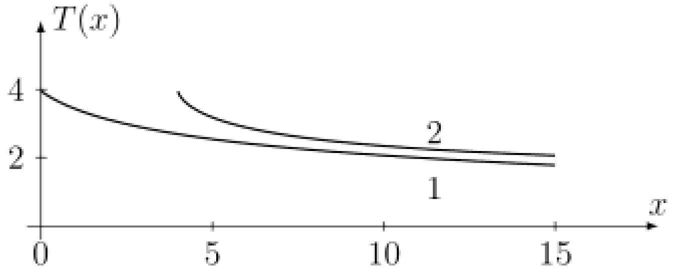

Figure 5. Period variables for function $\sin x^{4/3}$: $T_{4/3}(x),\ x \geq 0$ (graph 1);

$T^{-}_{4/3}(x),\ \ x \geq (2\pi)^{3/4} \approx 3.968$ (graph 2)

Correlation between trigonometric functions with constant and variable periods. From formulas (3a) and (3b), which are the variable periods of functions $\sin x^{\alpha}$ and $\cos x^{\alpha}$, in the partial case when $\alpha = 1$, we have $T(x) = -x + (x + 2\pi) = 2\pi$, $T^{-}(x) = x - (x - 2\pi) = 2\pi$, that is, we obtain the periods of the functions $\sin x$ and $\cos x$. Similarly, from formulas (5a) and (5b), we obtain that for $\alpha = 1$ the values of periods $T(x) = -x + (x + \pi) = \pi$, $T^{-}(x) = x - (x - \pi) = \pi$, which are the periods of functions tgx and $ctgx$. From these results, it can be seen that the periods of trigonometric functions $\sin x, \cos x$, $tgx, ctgx$, are partial case of variable periods for functions $\sin x^{\alpha}, \cos x^{\alpha}$, tgx^{α}, $ctgx^{\alpha}$ at $\alpha = 1$. Thus, functions $\sin x, \cos x$, $tgx, ctgx$ are **partial case** of functions $\sin x^{\alpha}, \cos x^{\alpha}$, tgx^{α}, $ctgx^{\alpha}$.

It should be noted that in the notation of variable periods, sometimes we will add index α, i.e., instead of $T(x)$ and $T^{-}(x)$ we will use notation $T_{\alpha}(x)$ and $T^{-}_{\alpha}(x)$, indicating that the period depends on the power index α .

Historical reference about the lack of scientific researches concerning the theory of periodic signals with variable period.

1. The above references were made only to single papers, which usually raises certain reservations, which can be summarized in the words: «Have the issues of the theory of signals and functions with variable period been considered anywhere else?».

The author of this paper has been and is almost always asked similar questions and doubts at conferences and scientific seminars. These questions could be followed by the opponents, own views, among which, for some reason, the question and answer often sounded: «Aren't almost periodic functions one of the types of periodic functions with variable period?» In private discussions, one could also hear: «What is a variable period?», «Can period be variable?», Didn't we even have the terms **variable period, periodic functions with variable period**? There were also other statements, all of which I cannot recall.

2. The string oscillation dispute. To answer the above mentioned questions, we analyzed the literature, going back to the time of the famous problem known as the «string oscillation disput». The best mathematicians of Europe at the time (the middle of the eighteenth century, when the theory of mechanical vibrations was emerging) took part in the dispute. These were I. Bernoulli (1667–1748), Taylor (1685–1731), D. Bernoulli (1700–1782), and D'Alembert (1717–1783), Euler (1707–1783), Lagrange (1736–1813), and others. The dispute was about the concept of the function, the arbitrary function. The mathematicians, especially D'Alembert and Euler, differed on these terms. The essence of this disagreement, as presented by Luzin [6, p. 38] or, for example, Fichtenholz [8, pp. 430–432], is as follows.

In order for a string with length l, to vibrate and thus generate a certain sound, first it must be deflected from its equilibrium position and be given certain shape in the form of **«arbitrary function»** $u_0(x)$. What is arbitrary function? For D'Alembert, **"arbitrary function»** $u_0(x)$ was **«arbitrary analytical expression»** written on the interval $[0,l]$ with only one formula. For Euler, the function could be even **«arbitrarily drawn curve»,** i.e., it could be defined graphically. At the same time, Euler did not require that the curve could be defined over the entire interval $[0,l]$ by single analytical expression. Which is **broader or narrower**? This question has led to the longstanding debate.

The discussion did not end there [9, p. 440, 441]. D. Bernoulli (1753) came up with a new solution to the problem. Based on physical considerations and proceeding from the fact that the **sound** of the string is formed by **main tone** and infinite number of **overtones** (an overtone [from German ober – high and Latin tonus – sound] is a component tone of the main sound that has higher frequency of **oscillation** than the main sound), he concluded that the vibrations of the string consisted of infinite number of different vibrations. Therefore, the shape of the string is formed by **superimposing sinusoids** (corresponding to different overtones**), the periods** of which decrease inversely with the natural numbers. In this case, the image of «**arbitrary function** $u_0(x)$» is represented by the equation

$$u_0(x) = a_1 \sin\frac{x}{l} + a_2 \sin\frac{2x}{l} + ... + a_n \sin\frac{nx}{l} + ... \qquad 6$$

In this way, **Bernoulli** came to the fundamental discovery, arguing [5, p. 178; 6, pp. 41–43] that any function can be expanded into trigonometric series, later called the **Fourier series.**

But both D'Alembert and Euler rejected Bernoulli's proposal. The main objection against Bernoulli was the **lack** of the rule for calculating the coefficients of trigonometric series (6).

The question of whether the analytical expression or curve is broader or narrower remained unresolved until the publication of J. Fourie's (1768–1830) papper on the theory of thermal conductivity (1822). He providedthea formula for determining the coefficients in Bernoulli series:

$$a_n = \frac{1}{\pi l}\int_{-l}^{l} u_0(x) \sin nx dx,\ n = 1,2,...,$$

which was immediately called «Fourier **coefficients**».

In general, as noted in [9, p. 452], Fourier showed that functions of a fairly wide class, including those determined in different parts by different analytical expressions, i.e., «discontinuous» according to Euler, can be represented on any finite interval by **trigonometric series**, i.e., analytically.

Fourier series are periodic functions. Following the modern representation, the Fourier series of function $f(x),\ x \in [0,T]$, can be written in the following form

$$\frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos k\omega_1 x + b_k \sin k\omega_1 x = \frac{a_0}{2}+\sum_{k=1}^{\infty} a_k \cos \omega_k x + b_k \sin \omega_k x, \qquad 7$$

where $\omega_1 = \frac{2\pi}{T}$ is main frequency, $k\omega_1 = k\frac{2\pi}{T} \overset{df}{=} \omega_k$ are multiples of frequencies that correspond to periods $T_k = \frac{T}{k}$. It can be seen that since for each term of series (7), in addition to the main period $\frac{T}{k}$, the period is also number T, the **Fourier series is the periodic** function with period T. We have not found any works that deal with **Fourier series with variable period.**

Thus, as can be seen from the history of the famous «string oscillation dispute», when two extremely important issues were initiated and **partially** resolved – the formation of the concept of the function and the expansion (representation) of the function in the form of Fourier series, as well as from all subsequent works, for example, [11–14], where various issues of Fourier series are considered, **the existence of periodic functions with a variable period is not even mentioned.**

3. Almost periodic functions. A separate branch of the Fourier series is **almost periodic functions** [11], i.e., functions that have the form of Fourier series, but among the sinusoidal terms of the series there are at least two terms whose “frequency arguments” are in irrational ratio. We are not talking about periodicity with variable period here either.

4. Exceptional examples of functions with variable period. In the literature, there are cases when, considering certain questions of the theory, functions with variable period «occur». Such, for example, are function $\sin\frac{1}{x}$, Chebyshov polynomials of the first kind $T_n(x) = \cos(n \arccos x),\ \ x \in [-1,1], n \geq 3.$ and function $\sin\frac{1}{x}$ is considered as an example of discontinuous function at point $x = 0$. Chebyshev polynomials are one of the systems of **classical orthogonal polynomials**. There is no information about the variable period of these functions in the literature. Note that in the following papers, we plan to consider some of the properties of these functions from the point of view of periodic functions with variable period.

Based on the search for investigations on the topic under consideration, it is clear that neither the famous «dispute» over the concept of function nor the theory of Fourier series, which are periodic functions, contains any mention of periodic functions with variable period. This is the reason to assert **that the issues of the theory of periodic functions with variable period were first introduced in the works of the author of this article.**

Conclusions. It is emphasized in tthis paper that in addition to periodic signals, the period of which is considered constant without any reservations, in practice there are signals for which periodicity is observed, but the period after which their values are repeated is no longer constant, but changes in a certain way. In-depth analysis of the literature has shown that **no theory** or analytical methods for studying such signals **have existed** until recently. It is pointed out that the first studies of signals with variable period were initiated in the works of the author of this paper. Some of the main results of these studies are presented, in particular: a) The class of periodic functions with variable period $T(x)$ is defined, some properties of variable period are considered, and variable period $T^{-}(x)$ is introduced for the case when the function argument is decreasing; b) Examples of the analytical formulation of such functions in the form of trigonometric functions with variable period $\sin x^{\alpha}, \cos x^{\alpha}$, $tgx^{\alpha}, ctgx^{\alpha}$ $\alpha > 1, \alpha \neq 1$, are given, and their variable periods are written. It is shown that the periods of trigonometric functions $\sin x, \cos x$, $tgx, ctgx$, are partial case of variable periods for functions $\sin x^{\alpha}, \cos x^{\alpha}$, $tgx^{\alpha}, ctgx^{\alpha}$, at $\alpha = 1$, and hence functions $\sin x, \cos x$, $tgx, ctgx$ are partial case of functions $\sin x^{\alpha}, \cos x^{\alpha}$, $tgx^{\alpha}, ctgx^{\alpha}$. The existence of functions $\sin x^{\alpha}, \cos x^{\alpha}$ raises the question of their use as basic functions to obtain trigonometric system of functions with variable period and, in the case of its orthogonality, to consider the construction of Fourier series of functions with variable period. It is planned to consider these issues in the following publications.

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УДК 517.51

ПЕРІОДИЧНІ ФУНКЦІЇ ЗІ ЗМІННИМ ПЕРІОДОМ – ОСНОВНІ ПОНЯТТЯ ТА ДЕЯКІ РЕЗУЛЬТАТИ ЇХ ДОСЛІДЖЕННЯ

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Резюме. *Одним із важливих прикладних напрямків математики є дослідження реальних сигналів. За своїми властивостями сигнали бувають досить різноманітними, відповідно різними є і методи їх дослідження. Серед цього різноманіття значну долю складають періодичні сигнали, але при цьому період, через який їх значення повторюються, вже не є постійним, а певним чином змінюється. Яскравим прикладом таких сигналів є електрокардіограми, отримані під час чи після дії на організм людини певного збудника спокою, наприклад, фізичного навантаження. На інтервалах часу, протягом яких пульс приходить у «норму», період електрокардіограми змінюється. Подібною до електрокардіограми буде поведінка спірограми, теж отриманої після дії навантаження чи іншого збудника психофізичного стану людини. Приклади аналогічних сигналів можна також навести із багатьох технічних систем. Аналіз літературних джерел показує, що для такого роду сигналів будь якої теорії та аналітичних методів їх дослідження до недавніх пір не існувало. Вказується, що перші дослідження сигналів зі змінним періодом започатковані в роботах автора цієї статті. Наведено деякі з основних результатів цих досліджень. Найперше, визначено клас періодичних функцій зі змінним періодом* $T(x)$. *Розглянуто деякі властивості змінного періоду, а також введено змінний період* $T^{-}(x)$ *для випадку, коли аргумент функції спадає. Наведено приклади аналітичного задавання таких функцій у вигляді тригонометричних функцій зі змінним періодом* $\sin x^{\alpha}, \cos x^{\alpha}$, $tgx^{\alpha}, ctgx^{\alpha}$ $\alpha > 1, \alpha \neq 1$, *та записано їх змінні періоди. Показано, що періоди тригонометричних функції* $\sin x, \cos x$, $tgx, ctgx$ *є частинним випадком змінних періодів для функцій* $\sin x^{\alpha}, \cos x^{\alpha}$, $tgx^{\alpha}, ctgx^{\alpha}$ *при* $\alpha = 1$, *а, отже, і функції* $\sin x, \cos x$, $tgx, ctgx$ *є частинним випадком функцій* $\sin x^{\alpha}, \cos x^{\alpha}$, $tgx^{\alpha}, ctgx^{\alpha}$. *Наявність функцій* $\sin x^{\alpha}, \cos x^{\alpha}$ *наштовхує на питання щодо їх використання як базових функцій для отримання тригонометричної системи функцій зі змінним періодом, і у випадку її ортогональності розглянути питання побудови рядів Фур'є функцій зі змінним періодом. До поставленого завдання запланованого звернутися в наступних публікаціях.*

Ключові слова: *періодичні функції зі змінним періодом, змінний період, тригонометричні функції зі змінним періодом, ряди Фур'є функцій зі змінним періодом.*

https://doi.org/10.33108/visnyk_tntu2024.01.046 *Отримано 03.01.2024*