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SOLUTIONS OF THE THEORY OF THERMOELASTICITY AND THERMAL CONDUCTIVITY IN THE CYLINDRICAL COORDINATE SYSTEM FOR AXISYMMETRIC TEMPERATURE

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Summary. *The paper uses the system of Navier equations in the stationary case. A cylindrical coordinate system is considered, when the temperature does not depend on the angular variable. A partial solution of the system of Navier equations, which does not contain elastic displacements, is called a purely temperature solution. It was established that for purely temperature solutions the sum of normal stresses is zero and the* volume deformation *is equal* $e = 3\alpha T$ *. An analytical expression of purely temperature displacements and stresses in the cylindrical coordinate system in the axisymmetric case was found. The solution of the boundary value problem of thermal conductivity, when the cylinder is heated on one end, cooled by liquid on the other with known heat losses on the side surface, is proposed. The solution of the boundary value problem of thermal conductivity for such a cylinder is given in the form of the sum of the basic temperature, which describes the heat balance, and the perturbed temperature. The basic temperature has a polynomial form and integrally satisfies the boundary conditions. The perturbed temperature has an exponential decrease with distance from the heated end and does not carry out integral heat transfer. The found dependencies were used and a new solution to the heat conduction equation was written in a cylindrical coordinate system. Simple formulas for expressing temperature changes have been obtained. A new temperature solution to the system of thermoelasticity equations in a cylindrical coordinate system has been written, when the temperature does not depend on the angular variable.*

Key words: *cylindrical coordinate system, thermoelastic state of the body, the axisymmetric temperature state, temperature stresses and displacements.*

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Introduction. Thermoelastic materials, under the influence of various temperature fields, are used in aerospace engineering and other technologies [1–3]. Elastic bodies, subjected to mechanical and thermal loads, are widely used in power engineering, technological, and engineering structures. [4, 5].

Review of known static solutions of the equations of elasticity theory and thermoelasticity. The methods for solving static boundary problems in elastic three-dimensional bodies primarily rely on constructing and utilizing various representations of the general solution to the equations of elasticity theory [1, 2, 5], where the temperature value is predetermined. When investigating three-dimensional static problems in axisymmetric thermoelasticity theory [1, 3, 4], some well-known solutions of elasticity theory equations are applied, with a specific temperature distribution added, determined by the thermoelastic potential [2, 4]. Many solutions to thermoelasticity theory problems are constructed using the thermoelastic potential [4–6]. For instance, in the article [7], some analytical solutions to a planar thermoelasticity problem are proposed using generalized functions and Fourier transformations, while in [8], the thermoelastic potential is used to reduce the three-dimensional thermoelastic state of a plate to solving a two-dimensional boundary problem. In the work [9], a methodology for the analytical-numerical assessment of temperature stresses in a finite-length hollow cylinder is developed based on the direct integration method. However, in the work

[10], some new physically justified partial solutions to the thermoelasticity theory in Cartesian coordinate systems have been found without using the thermoelastic potential. These solutions more accurately account for the influence of the temperature field on the stressed state of thermoelastic bodies. Furthermore, based on the work [11], a new representation of the general solution to the thermoelasticity equations through four harmonic functions has been proposed in the paper [12]. Physically justified temperature solutions to the Navier equations in a cylindrical coordinate system were found in [12], when the temperature does not depend on the axial variable.

The purpose of the article. To find purely temperature solutions of the Navier equations in a cylindrical coordinate system for axisymmetric temperature distribution.

1. Statement of the problem and formulation of equations of static thermoelasticity. Let's consider the general formulation of a three-dimensional static problem in thermoelasticity theory for a cylindrical isotropic body in the case of axisymmetric distribution of temperature and stresses. We set the initial temperature, when temperature stresses are zero, equal to zero. The temperature in the body varies within limits such that the elastic and thermal conductivity coefficients of the material can be modeled as constants.

We will use the Duhamel-Neumann relationship for thermoelastic stresses [1–3] in a homogeneous solid body in the axisymmetric case

$$\sigma_r = 2G\left[\varepsilon_r + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T\right], \ \sigma_\varphi = 2G\left[\varepsilon_\varphi + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T\right],$$

$$\sigma_z = 2G\left[\varepsilon_z + \frac{\nu}{1-2\nu}e - \frac{1+\nu}{1-2\nu}\alpha T\right], \ \tau_{r\varphi} = \tau_{\varphi z} = 0, \ \tau_{rz} = G\gamma_{rz}, \qquad (1)$$

where $G = E/2(1+\nu)$, E are shear and Young's moduli, u_r, $u_\varphi = 0$, u_z are elastic displacements, $\varepsilon_r = \frac{\partial u_r}{\partial r}$, $\varepsilon_\varphi = \frac{u_r}{r}$, $\varepsilon_z = \frac{\partial u_z}{\partial z}$ are deformations of relative elongation, $\gamma_{rz} = \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}$, $\gamma_{r\varphi} = 0$, $\gamma_{z\varphi} = 0$ are relative shear strains, $e = \varepsilon_r + \varepsilon_z + \varepsilon_\varphi$ – volume deformation, ν is Poisson's ratio, α is the coefficient of thermal expansion.

Let's substitute the relationship (1) into the equilibrium equation of a thermoelastic body in cylindrical coordinate system and write down the system of Navier differential equations for elastic displacements in the axisymmetric case [1, 2]

$$\Delta u_r + \frac{1}{1-2\nu}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} = 2\frac{1+\nu}{1-2\nu}\alpha\frac{\partial T}{\partial r},$$
$$\Delta u_z + \frac{1}{1-2\nu}\frac{\partial e}{\partial z} = 2\frac{1+\nu}{1-2\nu}\alpha\frac{\partial T}{\partial z}, \qquad (2)$$

where $\Delta = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ is the axisymmetric Laplace operator [6, 13].

We will consider that within the body, there is a stationary temperature field without internal heat sources, which satisfies the Laplace equation.

$$\Delta T(r,z) = 0. \qquad (3)$$

We will consider the system of equations of thermoelasticity theory (2) as a system of differential equations with known non-zero right-hand sides, determined by the harmonic temperature (3). The general solution of the system of equations (2) will be presented as a sum of homogeneous and particular solutions. It is known that the particular solution, as a rule, is not uniquely determined, but up to an unknown homogeneous solution.

When solving many practical and scientific problems of thermoelasticity, there arises a necessity to find a uniquely determined stress-strain state that depends solely on temperature and does not contain elastic displacements. Therefore, we will make the corresponding separation of displacements in the Cartesian coordinate system

$$u_j = u_j^e + u_j^\tau, \; j = \overline{1,3}, \tag{4}$$

where u_j^e are the components of the vector of elastic displacements (with index e), u_j^τ are the components of the temperature displacement (with index τ), which are definitely determined by temperature (3) and do not involve any elastic displacements.

Definition. *A partial solution u_r^τ, u_z^τ of the system of equations (2) in a thermoelastic medium or body will be referred to as purely temperature if it does not contain elastic displacements and is determined by a specified temperature T.*

A thermoelastic body differs from a medium in that when calculating the final stresses in the body, it is necessary to take into account the influence of purely temperature displacements and stresses that arise on the surface of the body due to the influence of the specified temperature T.

In the works [10, 12], known partial solutions of the Navier equations system are analyzed in Cartesian and cylindrical coordinate systems, which do not contain elastic displacements. Let's use the regularities found there and dependency (4) to obtain the following simple expressions:

$$\sigma_r^\tau + \sigma_\varphi^\tau + \sigma_z^\tau = 0, \;\; \mathrm{e}^\tau = \varepsilon_r^\tau + \varepsilon_\varphi^\tau + \varepsilon_z^\tau = 3\alpha T. \tag{5}$$

Let's introduce the value of volume deformation (5) into the system (2) and obtain two unconnected equations

$$\Delta \mathrm{u}_r^\tau - \frac{\mathrm{u}_r^\tau}{r^2} = -\alpha \frac{\partial T}{\partial r}, \;\; \Delta \mathrm{u}_z^\tau = -\alpha \frac{\partial T}{\partial z}. \tag{6}$$

In the work [10], the temperature solution of the Navier equations system has been found in the Cartesian coordinate system: $x_1 = x,\ x_2 = y,\ x_3 = z$

$$u_j^\tau = \frac{\partial \vartheta}{\partial x_j} + \chi_1 \Omega_j, \; j = \overline{1,3}, \tag{7}$$

where T, $\Omega_j = \int \mathrm{T} dx_j$ are three-dimensional harmonic functions, $\vartheta(x, y, z) = \chi(x\Omega_1 + y\Omega_2 + z\Omega_3)$ is biharmonic main function, $\chi = -\alpha/6$, $\chi_1 = 4\alpha/3$. The integrals Ω_j are defined such that they are harmonic functions, which equal zero when the temperature is zero. Solution (7) is obtained under the assumption that the temperature depends on three coordinates.

Due to the invariance of the gradient vector and in accordance with works [10, 12], the temperature solution of the system of equations (6) will be sought in the following form:

$$u_r^\tau = \frac{\partial \vartheta}{\partial r} + 2\chi_1 \Omega_r, \quad u_z^\tau = \frac{\partial \vartheta}{\partial z} + \chi_1 \Omega_z, \tag{8}$$

where $\vartheta(r,z) = \chi(2r\Omega_r + z\Omega_z)$ is biharmonic main function, $\Omega_z = \Omega_3 = \int T dz$ – satisfies the equation (3), Ω_r is an unknown function from temperature $T(r,z)$. From these definitions, it follows that functions $r\Omega_r, z\Omega_z$ are biharmonic ones, and the following equation is true

$$\Delta\vartheta(r,z) = -\alpha T(r,z). \tag{9}$$

Let's note that the solution (8) is obtained under the assumption that the temperature depends on two coordinates. If the temperature depends only on the radial coordinate, then the temperature and displacement will have the form

$$T(r) = c_0 + c_1 \ln r, \; u_z^\tau = \alpha z T(r), \; u_r^\tau = r(a_0 + a_1 \ln r),$$

where $a_1 = \alpha c_1$, $a_0 = \frac{1}{2}\alpha(2c_0 - c_1)$.

2. Constructing a partial solution of equations of the theory of thermoelasticity in cylindrical coordinate system for axisymmetric temperature.

Having substituted into the second equation (6) the displacement u_z^τ (8), and having taken into account the dependence (9), we saw that the equation was satisfied. Having substituted into the first equation (6) the displacement (8), we obtained the following equation

$$\Delta \frac{\partial \vartheta}{\partial r} - \frac{1}{r^2}\frac{\partial \vartheta}{\partial r} + 2\chi_1 \Delta\Omega_r - \frac{2\chi_1}{r^2}\Omega_r = -\alpha \frac{\partial T}{\partial r}. \tag{10}$$

Having taken into account the expression of the Laplace equation (3) and after some mathematical transformations [13], we wrote for the function $U(r,z)$ the formula of operators' permutation

$$\Delta \frac{\partial}{\partial r} U = \frac{\partial}{\partial r}\Delta U + \frac{1}{r^2}\frac{\partial}{\partial r} U. \tag{11}$$

To solve the equation (10) we have used the formulae (9), (11) and obtained the following ratio

$$\Delta \frac{\partial}{\partial r}\vartheta = \frac{1}{r^2}\frac{\partial}{\partial r}\vartheta - \alpha \frac{\partial}{\partial r} T. \tag{12}$$

Let's substitute the dependence (12) into the equation (10) and simplify it

$$\Delta\Omega_r - \frac{1}{r^2}\Omega_r = 0. \tag{13}$$

The direct verification has found that the solution of equation (13), which is consistent with the expression of functions (8), is expressed in terms of temperature as follows:

$$\Omega_r = \frac{1}{r}\int r\mathrm{T}dr.$$

Let's write down all integrals involved in the temperature solution (8) of the system of equations (6)

$$\Omega_r = \frac{1}{r}\int r\mathrm{T}dr,\ \Omega_z = \int Tdz,\ \vartheta(r,z) = \chi(2\int r\mathrm{T}dr + z\Omega_z) \qquad (14)$$

Therefore, the purely temperature solution of the axisymmetric equations of thermoelasticity (2) takes the form (8), (14).

3. Solving the equation of heat conductivity (3) and finding the axisymmetric distribution of temperature in the cylinder. Let's consider the cylinder: $D = \{(r,\varphi,z) \in ([0,R]\times[0,2\pi]\times[h_1,h_2])\}$, where $h_1 = 0$, $h_2 = h$, $h/R > 7$. Let's assume that the first ($z = 0$) end face of the cylinder is being heated, while the second ($z = h$) end face is subjected to intense cooling by a fluid, so that the temperature of the fluid near the end face remains constant. Heat exchange is specified on the end faces of the cylinder according to Newton's cooling law [2, 4]

$$\frac{\partial T(r,0)}{\partial z} = -\mu_1[\theta_1(r) - T(r,0)], \qquad (15)$$

$$\frac{\partial T(r,h)}{\partial z} = \mu_2[\theta_2^c - T(r,h)], \qquad (16)$$

where $\mu_j = \kappa_j/\lambda$, κ_j, $j = \overline{1,2}$ are the heat transfer coefficients on the cylinder's ends, λ is the coefficient of heat conductivity of the cylinder material, θ_j^c, $j = \overline{1,2}$ are the known average values of the environment temperature near the ends of the cylinder, $\theta_1(r) > T(r,0)$. Let's consider the heat flux density vector $\mathbf{q}$ [1, 4]. A constant component of the heat flux density vector $\mathbf{q}_r \mathrm{e}_r$ is specified on the lateral surface of the cylinder, directed normal to the lateral surface towards decreasing temperature. Therefore, the cylinder will be cooled on its lateral surface

$$\mathrm{q}_r = -\lambda \frac{\partial T(r,z)}{\partial r}\Big|_{r=R}, \qquad (17)$$

where q_r is a known constant amount of heat flows through an element of the cylindrical surface per unit time.

To solve the boundary problem (3), (15)–(17), we will proceed similarly to solving boundary problems in elasticity theory [11]. We will divide the overall temperature in the cylinder into two components. The solution to this problem will be sought in the form of a sum of the base temperature $T_0(r,z)$, which describes the heat balance, and the disturbed temperature $\mathrm{T}^p(r,z)$, which accurately accounts for the temperature $\theta_1(r)$. The introduced functions must satisfy the equation (3). However, the base temperature $T_0(r,z)$ has a polynomial form and precisely satisfies condition (17) and the integral boundary conditions (15), (16). It describes the heat transfer process in the cylinder, hence we refer to it as the base temperature. The disturbed temperature $\mathrm{T}^p(r,z)$ exponentially decreases with distance from the heated end. The integral heat transfer by temperature $\mathrm{T}^p(r,z)$ is equal to zero. Due to the condition $h/R>7$, it is practically equal to zero on the second end of the cylinder. The sum of temperatures $T_0(r,z)+T^p(r,z)$ must satisfy the conditions (15) – (17) completely.

To find the introduced functions, let's average the boundary problem (3), (15), (16) over the circular cross-section of the cylinder D, which has an area $S=\pi R^2$. We have integrated the Laplace equation (3) over the circular cross-section of the cylinder z and obtained the equations

$$\frac{\partial^2}{\partial z^2}\tilde{T}(z)=-\frac{2}{R}\frac{\partial}{\partial r}T(r,z)\,|_{\mathrm{r=R}}\,, \tag{18}$$

where $\tilde{\mathrm{T}}(z)=\frac{1}{S}\int\limits_S rT(r,z)dr$. Let's mark the average value of the temperature with a wave at the top. Let's also average relations (15), (16)

$$\frac{\partial\tilde{\mathrm{T}}(0)}{\partial z}=-\mu_1[\theta_1^c-\tilde{T}(0)], \tag{19}$$

$$\frac{\partial\tilde{T}(h)}{\partial z}=\mu_2[\theta_2^c-\tilde{T}(h)], \tag{20}$$

where $\theta_1^c=\frac{1}{S}\int\limits_S r\theta_1 dr>\theta_2^c$ is a known average value of the environment temperature near the first end of the cylinder.

Now we are finding the amount of heat [1, 4] entering the cylinder through the section $z=0$

$$\mathrm{q}_z(0)=-\lambda\frac{\partial}{\partial z}\tilde{\mathrm{T}}(z)\,|_{z=0}\,. \tag{21}$$

Let's imagine a portion of the cylinder $D_z=\{(r,z)\in([0,R]\times[0,z])\}$. We will find the amount of heat q_z that flows per unit time through the cross-section of the cylinder z

$$\mathrm{q}_z(z)=-\lambda S\frac{\partial\tilde{T}(z)}{\partial z}\,. \tag{22}$$

We are also calculating the amount of heat $\mathrm{q}_R(z)$ flowing out per unit of time through the lateral surface of the cylinder D_z

$$\mathrm{q}_R(z)=-\lambda 2\pi R\int\limits_0^z\frac{\partial T(R,z)}{\partial r}dz=2\pi R\int\limits_0^z \mathrm{q}_r dz=2\pi R\mathrm{q}_r z\,. \tag{23}$$

We take the relations (21)–(23) into consideration and write the heat balance

$$q_z(0) = q_z(z) + q_R(z). \tag{24}$$

We will use the relations (18)–(20) and find the base and the averaged temperature which must satisfy the formulae (3), (21)–(24)

$$T_0(r, z) = [2z^2 - r^2] + t_1 z + t_0, \ \tilde{T}_0(z) = 2g_1 z^2 + t_1 z + t_0 - \frac{g_1}{2} R^2, \tag{25}$$

where $\tilde{T}_0(z) = \frac{1}{S}\int_S rT_0(r, z)dr$, g_1, t_1, t_0 are the unknown coefficients. The condition (17) is fulfilled provided

$$g_1 = q_r/(2\lambda R). \tag{26}$$

Let’s substitute the temperature (25) into the expressions (22)–(24), use the dependence (26), and find the values that are included in the temperature balance

$$q_z(0) = -\lambda S t_1, \ q_z(z) = -\lambda S(4g_1 z + t_1), \ q_R(z) = 4S\lambda g_1 z. \tag{27}$$

Let's incorporate the components (27) into the heat balance equation (24) and ensure that it is true. From equation (27) it follows that $t_1 < 0$.

Let’s substitute the temperature (25) into the averaged boundary conditions (19), (20) taking into account the dependence (26). We obtain a system of two equations to determine the coefficients t_1 and t_0

$$t_1 = -\mu_1[\theta_1^c - t_0], \quad t_1(1+\mu_2 h) = \mu_2(\theta_2^c - t_0 + \frac{g_1}{2} R^2) - 2g_1 h(2+\mu_2 h). \tag{28}$$

Let’s solve the equations (28) and find the coefficients of the base temperature (25)

$$t_0 = \frac{\mu_2\theta_2^c + g_1[\frac{\mu_2}{2} R^2 - 2h(2+\mu_2 h)] + (1+\mu_2 h)\mu_1\theta_1^c}{[\mu_2 + \mu_1(1+\mu_2 h)]}, \ t_1 = -\mu_1[\theta_1^c - t_0]. \tag{29}$$

Let's find the integrals (14) for the base temperature, which is determined by coefficients (25), (26), (29). Substituting these integrals into the relations (8), we will determine the purely temperature displacements for the base temperature (25)

$$u_r^\tau = \frac{1}{3}\alpha r g_1[7z^2 - r^2] + \alpha r(t_1 z + t_0), \ u_z^\tau = \alpha g_1[\frac{4}{3}z^3 - 3zr^2] + \frac{1}{2}\alpha t_1(z^2 - r^2) + \alpha t_0 z. \tag{30}$$

We must admit that the displacements (30) have been not causing any stresses.

As the base temperature $T_0(r, z)$ (25) has taken into account the conditions (16), (17), then the disturbed temperature $T^p(r, z)$ must satisfy the boundary conditions:

$$\frac{\partial T^{p}(r,z)}{\partial r}\Big|_{r=R}=0 \tag{31}$$

on the lateral surface of the cylinder,

$$\frac{\partial T^{p}(r,h)}{\partial z}=-\mu_2 T^{p}(r,h) \tag{32}$$

at the other end of the cylinder. We will seek the solution to equation (3) using the method of separation of variables in order to satisfy the condition (31). We are writing the disturbed temperature $T^{p}(r,z)$ as a series [13, 14]

$$T^{p}(r,z)=\sum_{k=1}^{\infty} a_k J_0(\beta_k r) e^{-\beta_k z}\,, \tag{33}$$

where a_k are the unknown coefficients, $\beta_k > 0$ are eigenvalues, which satisfy the equation

$$J_1(\beta_k R)=0 \tag{34}$$

and they are arranged in ascending order. Since $h/R>7$, $\beta_k>0$, the functions (33) decrease exponentially, they will satisfy condition (32) with high accuracy. It is known from [14], that the system of functions { $J_0(\beta_k r)$ }, $k=1,2,...$ by fulfilling the condition (34), it is orthogonal over the interval $[0,R]$ with the weight function r:

$$\int_0^R r J_0(\beta_k r) J_0(\beta_j r)\,dr=0,\ \ k\neq j\,. \tag{35}$$

Let’s take into account the averaged base temperature (25), the disturbed temperature (33) and write the condition (15)

$$\sum_{k=1}^{\infty} a_k \beta_k J_0(\beta_k r)=\mu_1[\theta_1(r)-\sum_{k=1}^{\infty} a_k J_0(\beta_k r)]\,. \tag{36}$$

We will take into account the conditions of orthogonality (35), and from the condition (36) we will find the coefficients a_k.

Using the known disturbed temperature (33), and employing formulas (8), (14), we will determine the purely temperature displacements. Adding to them the temperature displacements (30), we will express the total temperature displacements in the cylinder.

4. Discussion of the results. In the theory of thermoelasticity, the thermoelastic potential of displacements Φ is widely used [1, 4]

$$\mathbf{u}^{p}=\mathrm{grad}\,\Phi\,. \tag{37}$$

Displacements from the potential will be denoted by the letter «*p*». The function Φ is defined in such a way as to be a partial solution of the system of thermoelasticity equations. It determines the volume deformation and satisfies the Poisson equation:

$$e^{p} = \frac{1+\nu}{1-\nu}\alpha T \,,\ \nabla^{2}\Phi = \frac{1+\nu}{1-\nu}\alpha T \,. \tag{38}$$

As we can see, the expression for volume deformation (38) differs from the volume deformation of the purely temperature solution (5). This is because in the displacement equation (37), in addition to temperature components, elastic displacements are also included. As a result, the solution (37) will not accurately account for the influence of the temperature field on displacements and corresponding stresses in the thermoelastic cylinder.

Let's consider how displacements (8), (37) differ in the case of linear temperature $\mathrm{T}_1(z) = t_1 z + t_0$, for which temperature displacements are known [6]

$$u_r^{\tau} = \alpha r(t_1 z + t_0)\,,\ u_z^{\tau} = \frac{1}{2}\alpha t_1(z^2 - r^2) + \alpha t_0 z \,. \tag{39}$$

The temperature displacements (39) coincide with the linear part of displacements (30), which are determined by the temperature $\mathrm{T}_1(z) = t_1 z + t_0$.

First, let's construct a particular solution of the last equation (38). To do this, we'll express the linear temperature $\mathrm{T}_1(z)$ through the derivative

$$\mathrm{T}_1(z) = \frac{\partial \varphi(r,z)}{\partial z} \,,$$

where function $\varphi(rz) = \frac{t_1}{4}(2z^2 - r^2) + t_0 z$ satisfies the equation (3). It enables writing the solution of the last equation (38) as follows

$$\Phi = \frac{1+\nu}{1-\nu}\frac{\alpha}{2} z\varphi(r,z) \,. \tag{40}$$

Using ratios (37), (40) displacements is found

$$u_r^{p} = \frac{\partial \Phi}{\partial r} = -\frac{1+\nu}{1-\nu}\alpha\frac{t_1}{4} zr \,,\ u_z^{p} = \frac{\partial \Phi}{\partial z} = \frac{1+\nu}{1-\nu}\alpha[\frac{t_1}{8}(6z^2 - r^2) + t_0 z] \,. \tag{41}$$

The representation of displacements (41) significantly differs from purely temperature displacements (39). It is important to note that displacements (41) do not coincide with the physically justified temperature displacements (39) for both constant $\mathrm{T}(z) = t_0$, and linear $\mathrm{T}(z) = t_1 z$ temperature cases.

Conclusions. It has been discovered that for purely temperature displacements in the cylinder, the volume deformation is equal to $e = 3\alpha T$, and the sum of normal stresses equals to zero. In a cylinder heated on one end and cooled on the other with known heat losses on the lateral surface, the temperature is described by a quadratic base temperature and a disturbed temperature. The disturbed temperature exponentially decreases with distance from the heated end. Simple dependencies (8), (14) have been obtained to determine temperature displacements in the cylindrical coordinate system in the axisymmetric case. The found formulas allow solving problems related to determining the thermoelastic state of cylindrical bodies.

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РОЗВ'ЯЗКИ ТЕОРІЇ ТЕРМОПРУЖНОСТІ Й ТЕПЛОПРОВІДНОСТІ В ЦИЛІНДРИЧНІЙ СИСТЕМІ КООРДИНАТ ДЛЯ ОСЕСИМЕТРИЧНОЇ ТЕМПЕРАТУРИ

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Резюме. *Для описування термопружного стану використано лінійну статичну модель тривимірного ізотропного тіла під дією стаціонарного температурного поля. Розглянута модель деформованого тіла базується на поданні переміщень і напружень у циліндричній системі координат в осесиметричному випадку через гармонічні функції. Використано співвідношення Дюамеля–Неймана для подання термопружних напружень в однорідному твердому тілі. Розглянуто випадок, коли стаціонарна температура задовольняє рівняння Лапласа в циліндричній системі координат в осесиметричному випадку. Після підстановки термопружних напружень у рівняння рівноваги термопружного тіла отримано систему диференціальних рівнянь Нав'є в частинних похідних другого порядку на пружні й температурні переміщення. Загальний розв'язок неведено у вигляді суми однорідного й часткового розв'язку, який не містить пружних переміщень. Цей частковий розв'язок системи рівнянь Нав'є названо чисто температурним розв'язком. Переміщення, деформації й напруження, які визначаються цими температурними розв'язками, названі температурними. Використано фізичні й математичні особливості термопружного напруженого стану й показано, що для чисто температурних розв'язків сума нормальних напружень дорівнює нулю, а об'ємне розширення дорівнює* $e = 3\alpha T$. *Знайдено аналітичний вираз чисто температурних переміщень і напружень у циліндричній системі координат в осесиметричному випадку. Запропоновано розв'язок крайової задачі теплопровідності, коли циліндр підігрівається на одному торці, охолоджується рідиною на іншому з відомими тепловими втратами на бічній поверхні. Розв'язок крайової задачі теплопровідності для такого циліндра наведено у вигляді суми основної температури, яка описує баланс теплоти, і збуреної температури. Основна температура має поліноміальний вигляд і інтегрально задовольняє крайові умови. Збурена температура має експонентне спадання при віддалені від нагрітого торця й не здійснює інтегрального переносу теплоти. Використано знайдені залежності й записано температурний розв'язок системи рівнянь термопружності в циліндричній системі координат, коли температура не залежить від кутової змінної. Отримано прості формули для вираження температурних напружень. Побудовано загальний розв'язок рівнянь теорії термопружності через три гармонічні функції, коли температурне поле не залежить від осьової координати.*

Ключові слова: *циліндрична система координат, термопружний стан тіла, фізичні характеристики температурного стану, температурні напруження і переміщення.*

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