Вісник Тернопільського національного технічного університету
Scientific Journal of the Ternopil National Technical University
2016, № 2 (82)
ISSN 1727-7108. Web: ***visnyk.tntu.edu.ua***

UDC 539.3

CONTACT INTERACTION OF THE PARABOLIC PUNCH WITH PRELIMINARY STRESSED PLATE FIXED ON THE RIGID BASIS

Iryna Habrusieva; Oleh Panchuk; Borys Shelestovskyi

Ivan Puliuy Ternopil National Technical University, Ukraine

Resume*. Solution of the contact problem on the interaction of the punch with preliminary stressed thick plate is presented in the article. Analytical solutions for the plate were found while it's modeling by the preliminary stressed layer of the finale thickness. System of double and triple integral equations obtained here and solved while presenting the necessary functions as the finale set sums according to the Bessel functions with the unknown coefficients and further obtaining of the finale systems of linear algebraic equations for their finding. Basing on the obtained stress-strain state the effect of the punch configuration on the stressed state of the thick plate has been analyzed.*

Key words*: linear theory of elasticity, contact interaction, contact stresses, parabolic punch, plate, layer, initial deformation, preliminary stresse*

Received 05.05.2016

Problem setting. Determination of contact stresses and deformations in the interaction of rigid punch with elastic plate is an important objective in the design of machine parts and elements of buildings, especially when assessing the strength of reinforced concrete slabs, monolithic foundation slabs in construction, pavement etc. To minimize calculation errors maximum number of factors affecting the contact interaction should be taken into account. The initial deformations, which directly influence contact stresses and displacements, are among the key factors.

Analysis of the known research results. Many scientists, including domestic ones, studied the interaction of bodies with existing residual permanent deformations. In general, setting of such problems requires the involvement of nonlinear elasticity theory apparatus, but in dealing with rather large initial deformations, its linearized version can also be used [1].

Despite the increasing number of studies on the contact interaction of bodies in their pre-stressed state [2-4], the problem of pressure of a parabolic punch mounted on a rigid foundation of a pre-stressed hard thick plate is still not resolved for compressible and incompressible bodies in general with arbitrary structure of the elastic potential.

Problem setting. Setting and problem solving is done within linearized elasticity theory. Thick plate is modeled by an isotropic pre-stressed layer.

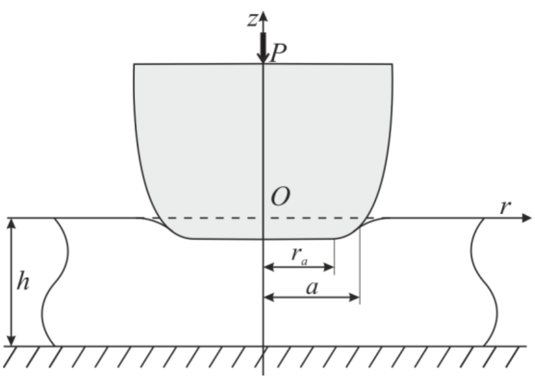

Figure 1. Contact interaction scheme of parabolic punch and layer

Consider a rigid parabolic punch that without rotation and friction goes forward and presses into a pre-strained layer with thickness h, mounted on a rigid base.

Let us assign a cylindrical coordinate system (O, r, θ, z) so that the coordinate plane (r, O, θ) coincides with the upper plane layer limit and axis Oz – with force application line P (Fig. 1).

Suppose contact area a radius is known and determine the corresponding focal parameter of the parabola R. Based on the formulation of the problem the function can be described as a graph rotation around the axis Oz which forms punch

$$W(r)=\begin{cases}0,\ 0\le r\le r_a;\\ \dfrac{1}{2R}(r-r_a)^2,\ r_a<r.\end{cases}$$

The boundary conditions of the problem appear to be as follows:

$$\sigma_{rz}(r,0)=0,\ 0\le r<\infty;\tag{1}$$

$$\sigma_{zz}(r,0)=0,\ a\le r;\tag{2}$$

$$u_z(r,0)=\omega(r),\ 0\le r\le a;\tag{3}$$

$$u_r(r,-h)=0,\ 0\le r<\infty;\tag{4}$$

$$u_z(r,-h)=0,\ 0\le r<\infty.\tag{5}$$

The function $\omega(r)$ describes the movement of the dots of elastic layer upper boundary plane in its contact area with a hard punch. Therefore, based on the type of function $W(r)$, it can be written:

$$\omega(r)=\begin{cases}\omega(a)-\dfrac{(a-r_a)^2}{2R},\ 0\le r\le r_a;\\ \omega(a)-\dfrac{1}{2R}\left[(a-r_a)^2-(r-r_a)^2\right],\ r_a<r\le a.\end{cases}\tag{6}$$

The solution of the problem. Let us assume that residual permanent stresses, present in the layer, are uniform. It makes possible to use the following expression for the components of the stress tensor and displacement vector [1]

$$\begin{gathered}\sigma_{rz}(r,z)=-c_{31}\int_0^\infty\alpha^3\left\{A_1sh(\alpha z)+A_2\left[s_0sh(\alpha z)+\alpha zch(\alpha z)\right]+\right.\\ \left.+B_1ch(\alpha z)+B_2\left[s_0\,ch(\alpha z)+\alpha z\,sh(\alpha z)\right]\right\}J_1(\alpha r)d\alpha;\\ \sigma_{zz}(r,z)=c_{33}\int_0^\infty\alpha^3\left\{A_1ch(\alpha z)+A_2\left[sch(\alpha z)+\alpha zsh(\alpha z)\right]+\right.\\ \left.+B_1sh(\alpha z)+B_2\left[ssh(\alpha z)+\alpha zch(\alpha z)\right]\right\}J_0(\alpha r)d\alpha;\end{gathered}\tag{7}$$

$$\begin{gathered}u_r(r,z)=-\int_0^\infty\alpha^2\left\{A_1ch(\alpha z)+A_2\left[ch(\alpha z)+\alpha z\,sh(\alpha z)\right]+\right.\\ \left.+B_1sh(\alpha z)+B_2\left[sh(\alpha z)+\alpha z\,ch(\alpha z)\right]\right\}J_1(\alpha r)d\alpha;\\ u_z(r,z)=m\int_0^\infty\alpha^2\left\{A_1sh(\alpha z)+A_2\left[s_1sh(\alpha z)+\alpha z\,ch(\alpha z)\right]+\right.\\ \left.+B_1ch(\alpha z)+B_2\left[s_1ch(\alpha z)+\alpha z\,sh(\alpha z)\right]\right\}J_0(\alpha r)d\alpha.\end{gathered}\tag{8}$$

Constants c_{31}, c_{33}, m, s, s_0, s_1 depend on the nature of the elastic potential and are selected separately in each case. [1] The function's unknowns A_1, B_1, A_2, B_2 are determined from the boundary conditions of the problem.

On the top boundary plane layer when $z = 0$ of the relations (7) – (8) we obtain

$$\sigma_{zz}(r,0) = c_{33}\int_0^\infty \alpha^3 \{A_1 + A_2 s\} J_0(\alpha r)d\alpha ; \tag{9}$$

$$\sigma_{rz}(r,0) = -c_{31}\int_0^\infty \alpha^3 \{B_1 + B_2 s_0\} J_1(\alpha r)d\alpha ; \tag{10}$$

$$u_z(r,0) = m\int_0^\infty \alpha^2 \{B_1 + B_2 s_1\} J_0(\alpha r)d\alpha . \tag{11}$$

The bottom boundary plane layer at $z = -h$ gives the following

$$\begin{aligned} u_r(r,-h) = -\int_0^\infty \alpha^2 \Big\{ & A_1 ch(\alpha h) + A_2\left[ch(\alpha h) + \alpha h\, sh(\alpha h)\right] + \\ & - B_1 sh(\alpha h) - B_2\left[sh(\alpha h) + \alpha h\, ch(\alpha h)\right]\Big\} J_1(\alpha r)d\alpha ; \end{aligned} \tag{12}$$

$$\begin{aligned} u_z(r,-h) = m\int_0^\infty \alpha^2 \Big\{ & -A_1 sh(\alpha h) + A_2\left[-s_1\, sh(\alpha h) - \alpha h\, ch(\alpha h)\right] + \\ & + B_1 ch(\alpha h) + B_2\left[s_1\, ch(\alpha h) + \alpha h\, sh(\alpha h)\right]\Big\} J_0(\alpha r)d\alpha . \end{aligned} \tag{13}$$

Demanding fulfillment of boundary condition (1) in the equality (10) we obtain the relation between functions B_1 and B_2

$$B_1 + B_2\, s_0 = 0;\ \Rightarrow\ B_1 = -s_0\, B_2 . \tag{14}$$

Substituting (14) in relations (12) – (13) and satisfying the boundary conditions (4) – (5), we will have a system of equations regarding unknowns A_1 and A_2

$$\begin{cases} A_1 ch(\alpha h) + A_2\left[ch(\alpha h) + \alpha h\, sh(\alpha h)\right] = B_2\left[\alpha h\, ch(\alpha h) + (1 - s_0) sh(\alpha h)\right]; \\ A_1 sh(\alpha h) + A_2\left[s_1\, sh(\alpha h) + \alpha h\, ch(\alpha h)\right] = B_2\left[(s_1 - s_0) ch(\alpha h) + \alpha h\, sh(\alpha h)\right]. \end{cases} \tag{15}$$

Upon solving (15), we obtain expressions for A_1 and A_2 by function B_2

$$\begin{aligned} A_1 &= \frac{(\alpha h)^2 - s_1 + s_0\, ch^2(\alpha h) - s_0 s_1\, sh^2(\alpha h)}{(s_1 - 1) ch(\alpha h) sh(\alpha h) + \alpha h} B_2; \\ A_2 &= \frac{(s_1 - s_0) ch^2(\alpha h) - (1 - s_0) sh^2(\alpha h)}{(s_1 - 1) ch(\alpha h) sh(\alpha h) + \alpha h} B_2. \end{aligned} \tag{16}$$

Given the relations (14) and (16) Expressions (9) and (11) take the following form:

$$\sigma_{zz}(r,0)=c_{33}(s-s_0)\int_0^\infty \frac{\alpha^3 B_2}{\Delta(\alpha)} J_0(\alpha r)d\alpha\,; \tag{17}$$

$$u_z(r,0)=m(s_1-s_0)\int_0^\infty \alpha^2 B_2 J_0(\alpha r)d\alpha\,; \tag{18}$$

$$\Delta(\alpha)=\frac{(s_1-1)sh(\alpha h)ch(\alpha h)+\alpha h}{(\alpha h)^2-s_1+(s_0+ss_1-ss_0)ch^2(\alpha h)-(s_0s_1+s-ss_0)sh^2(\alpha h)}.$$

Satisfying the boundary conditions (2) of (17) we will get

$$c_{33}(s-s_0)\int_0^\infty \frac{\alpha^3 B_2}{\Delta(\alpha)} J_0(\alpha r)d\alpha=0,\; a\le r. \tag{19}$$

Let us introduce an unknown function $x(r)$, $0\le r\le a$, by which continue correlation (19) for the period $0\le r<\infty$

$$c_{33}(s-s_0)\int_0^\infty \frac{\alpha^3 B_2}{\Delta(\alpha)} J_0(\alpha r)d\alpha = x(r)\eta(a-r), \qquad 0\le r<\infty\,, \tag{20}$$

where $\eta(r)$ – Heaviside function.

The function $x(r)$ determines the distribution of contact stresses under the punch. Taking into account their continuity and the lack of contact at the area boundary (at $r=a$), let us represent $x(r)$ as a segment of Fourier generalized series with functions $J_0\left(\frac{\lambda_n}{a}r\right)$

$$\sigma_{zz}(r,0)=x(r)=\sum_{n=1}^{N} a_n J_0\left(\frac{\lambda_n}{a}r\right),\; 0\le r\le a\,, \tag{21}$$

where λ_n, $n=\overline{1,N}$ – positive roots of Bessel function $J_0(\lambda_n)=0$; and a_n – unknown coefficients.

Applying the formula to convert Hankel transform into formula (20) and taking into account the representation (21), we get

$$\frac{\alpha^2 B_2}{\Delta(\alpha)}=\frac{1}{c_{33}(s-s_0)}\sum_{n=1}^{N} a_n \int_0^a r\,J_0\left(\frac{\lambda_n}{a}r\right)J_0(\alpha r)dr\,. \tag{22}$$

Entering designations

$$\Psi_n(\alpha)=\int_0^a rJ_0\left(\frac{\lambda_n}{a}r\right)J_0(\alpha r)dr\,,$$

from (22) we obtain

$$\alpha^2 B_2=\frac{\Delta(\alpha)}{c_{33}(s-s_0)}\sum_{n=1}^{N} a_n\,\Psi_n(\alpha). \tag{23}$$

Substituting equation (23) in equation (18), we obtain

$$u_z(r,0)=k_1\sum_{n=1}^{N}a_n\int_0^{\infty}\Delta(\alpha)\Psi_n(\alpha)J_0(\alpha r)d\alpha\ ,\ k_1=\frac{m(s_1-s_0)}{c_{33}(s-s_0)}. \quad (24)$$

Demanding fulfillment of boundary condition (3) and taking into account (24), we find

$$k_1\sum_{n=1}^{N}a_n\int_0^{\infty}\Delta(\alpha)\Psi_n(\alpha)\{J_0(\alpha r)-J_0(\alpha a)\}d\alpha=\omega^*(r), \quad (25)$$

$$\omega^*(r)=\begin{cases}-\dfrac{1}{2R}(r_a-a)^2,\ 0\le r\le r_a;\\ \dfrac{1}{2R}\left[(r_a-r)^2-(r_a-a)^2\right],\ r_a<r\le a.\end{cases}$$

Multiplying equation (25) by $rJ_0\left(\frac{\lambda_q}{a}r\right)$ and upon integrating the obtained expressions on r from 0 to a, we will have

$$\sum_{n=1}^{N}a_n\int_0^{\infty}\Delta(\alpha)\Psi_n(\alpha)\left[\Psi_q(\alpha)-K_qJ_0(\alpha a)\right]d\alpha=\frac{w_q}{k_1},\ q=\overline{1,N}\ ; \quad (26)$$

$$K_q=\int_0^a rJ_0\left(\frac{\lambda_q}{a}r\right)dr\ ;\ w_q=\int_0^a r\omega^*(r)J_0\left(\frac{\lambda_q}{a}r\right)dr\ .$$

Enter designation

$$a_n=\frac{1}{2Rk_1}a_n^*. \quad (27)$$

In view of which (26) we get a system of linear equations N concerning unknowns a_n^*.

The expression to determine the focal parameter R, which is a part of formula (27), we find from the condition of punch equilibrium

$$2\pi\int_0^a r\sigma_{zz}(r,0)dr=-P\ . \quad (28)$$

Substituting in (28) Expressions (21) and (27), we define

$$\frac{1}{2R}=\frac{k_1P}{2\pi}\frac{-1}{\sum_{n=1}^{N}a_n^*K_n}, \quad (29)$$

considering (29), (27) of (21) we will have a formula for determining the distribution of contact stresses under the punch

$$\sigma_{zz}(r,0) = -\frac{P}{2\pi}\frac{\sum_{n=1}^{N} a_n^* J_0\left(\frac{\lambda_n}{a}r\right)}{\sum_{n=1}^{N} a_n^* K_n}. \quad (30)$$

On the basis of relations (18) and (27) we find the formula for determining vertical displacements of the upper boundary plane layer

$$u_z(r,0) = -\frac{k_1 P}{2\pi}\frac{\sum_{n=1}^{N} a_n^* \int_0^\infty \Delta(\alpha)\Psi_n(\alpha)J_0(\alpha r)d\alpha}{\sum_{n=1}^{N} a_n^* K_n}. \quad (31)$$

The numerical example. Fig. 2 and 3 show the graphics of functions $\sigma^* = \frac{\sigma_{zz}(r,0)}{P}$ and $u^* = \frac{u_z(r,0)}{P}$ that characterize the distribution of contact stresses (30) and vertical displacements (31). As a numerical example, we consider the case of the presence of harmonic type elastic potential in the plate [1] and the presence of a flat section in the punch basis with $h=1$, $a=1$. Curve 1 corresponds to $r_a = 0$, curve 2 – to $r_a = 0.2$, and curve 3 – to $r_a = 0.5$.

As can be seen from the figures, the shape of the punch significantly influences the size and distribution of contact stresses. In particular, the parabolic punches without a flat area at their base have the extreme values of contact stresses in the center of contact area. The appearance of flat areas causes extremum points to shift to the edge of the contact area and reduces their absolute value.

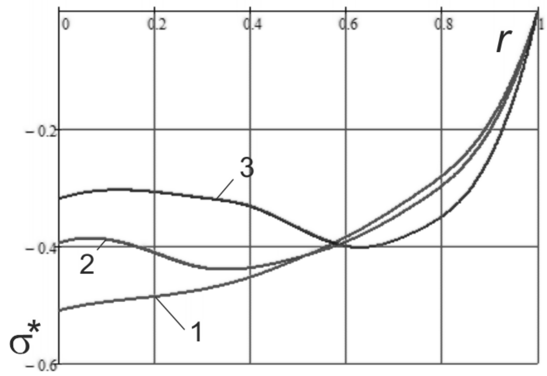

Figure 2. The distribution of contact stresses

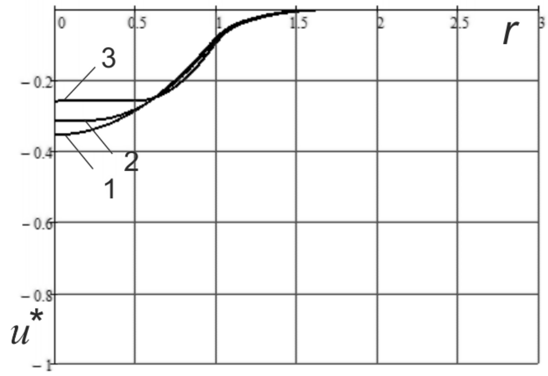

Figure 3. Vertical movement

Conclusions. Within the framework of linearized elasticity theory the authors presented formulation and solution of axis-symmetric contact problem of the interaction of a parabolic punch with a pre-stressed plate, mounted on a rigid base. The influence of flat areas in a rigid punch base has been analyzed. It was established that the shape of the punch influences the size and distribution of contact stresses. In particular, the parabolic punches without a flat area at their base have the extreme values of contact stresses in the center of the contact area. The appearance of flat areas causes extremum points to shift to the edge of the contact area and reduces their absolute value. If $r_a = 0.2a$ (Fig. 1), the absolute value of contact stress reduces by approximately 10%, and if $r_a = 0.5a$ – by 12%. Vertical displacements meanwhile decline by 9% and 11% respectively. Obtained results can be used in the development of contact stresses or vertical displacement reducing techniques when designing various kinds of structures.

References

1. Huz O.M. Kontaktna vzaiemodiia pruzhnykh til z pochatkovymy napruzhenniamy: Navch. posibnyk, Huz O.M., Babych S.Yu., Rudnytskyi V.B. K: Vyshcha shkola, 1995. P. 304.
2. Habrusieva I.Yu. Contact interaction of a circular punch with a preliminarily stressed isotropic layer, B.G. Shelestovs`kyi, I.Yu. Habrusieva, Journal of mathematical sciences. 2012. Vol. 186, no. 1. Pp. 48-60.
3. Habrusieva I.Yu. Kontaktna zadacha dlia parabolichnoho shtampa ta poperedno napruzhenoho pivprostoru, I.Yu. Habrusieva, Mizhvuzivskyi zbirnyk "Naukovi notatky". 2015. Vol. 51. Pp. 231-236.
4. Shelestovskyi B.H. Vzaiemodiia kiltsevoho shtampa iz poperedno napruzhenym sharom u vypadku potentsialu Bartenieva-Khazanovycha, B.H. Shelestovskyi, I.Yu. Habrusieva, Visnyk TNTU. 2010. Vol. 15, no. 3. Pp. 14-22.

Список використаної літератури

1. Гузь, О.М. Контактна взаємодія пружних тіл з початковими напруженнями: навч. посібник [Текст] / О.М. Гузь, С.Ю. Бабич, В.Б. Рудницький – К: Вища школа, 1995. – 304 с.
2. Gabrusseva, I.Yu. Contact interaction of a circular punch with a preliminarily stressed isotropic layer [Text] / B.G. Shelestovs`kyi, I.Yu. Gabrusseva // Journal of mathematical sciences. – 2012. – Vol. 186, no. 1. – P. 48 – 60.
3. Габрусєва, І.Ю. Контактна задача для параболічного штампа та попередньо напруженого півпростору [Текст] / І.Ю. Габрусєва // Міжвузівський збірник «Наукові нотатки». – 2015. – Вип. 51. – С. 231 – 236.
4. Шелестовський, Б.Г. Взаємодія кільцевого штампа із попередньо напруженим шаром у випадку потенціалу Бартенєва-Хазановича [Текст] / Б.Г. Шелестовський, І.Ю. Габрусєва // Вісник ТНТУ. – 2010. – Т. 15, № 3. – С. 14 – 22.

УДК 539.3

КОНТАКТНА ВЗАЄМОДІЯ ПАРАБОЛІЧНОГО ШТАМПА ІЗ ПОПЕРЕДНЬО НАПРУЖЕНОЮ ПЛИТОЮ, ЗАКРІПЛЕНОЮ НА ЖОРСТКІЙ ОСНОВІ

Ірина Габрусєва; Олег Панчук; Борис Шелестовський

Тернопільський національний технічний університет імені Івана Пулюя, Тернопіль, Україна

Резюме. *Наведено розв'язок контактної задачі про взаємодію параболічного штампа із попередньо напруженою товстою плитою. Побудова аналітичних розв'язків для плити проводиться шляхом її моделювання попередньо напруженим шаром скінченної товщини. Системи парних та потрійних інтегральних рівнянь, що при цьому отримуються, розв'язуються за допомогою подання шуканих функцій у вигляді скінченних сум ряду за функціями Бесселя з невідомими коефіцієнтами та подальшим отриманням скінчених систем лінійних алгебраїчних рівнянь для їх знаходження. На основі отриманого напружено-деформівного стану проаналізовано вплив форми штампа на напружений стан товстої плити.*

Ключові слова: *лінеаризована теорія пружності, контактна взаємодія, контактні напруження, параболічний штамп, плита, шар, початкові деформації, попередні напруження.*

Отримано 05.05.2016