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RESEARCH OF CONTACT INTERACTION OF PRESTRESSED STAMPS, LAYER AND FOUNDATION WITHOUT FRICTION

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Summary. *Solution to the problem of contact pressure of pre-stressed cylinder, layer and the foundation without friction is presented within linear elasticity theory. The research is presented in general terms for the theory of large initial deformations and for two versions of small initial deformations theory in the random structure of elastic potential.*

Key words*: linear elasticity theory, initial (residual) stress, method of successive approximations reduction method, Fredholm integral equations*

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Type Codes:

E – elasticity modulus of the 1st type;

λ_i – elongation factors that determine the movement of the original (remaining) state;

y_i – coordinates of the initial strain state;

x_i – Lagrange coordinates;

h_1 – layer thickness in the initial deformed state;

h_2 – layer thickness in the unstrained state;

δ_{im} – Kronecker symbol;

$\omega'_{im\alpha\beta}$, $\kappa'_{im\alpha\beta}$ – components of elastic moduli tensor of the fourth order;

$J_v(x), I_v(x)$ – – Bessel functions of real and imaginary argument.

Introduction. Contact problems make an important section of the mechanics of solid deformable body and form the theoretical basis for the calculation of the contact strength, rigidity and durability of mobile and fixed connections.

Applied needs of science, modern technology and new technologies created the need to predict contact behavior of various structures, and in recent decades stimulated the development of various mathematical models and methods of contact mechanics of bodies with different properties.

One of the important factors influencing contact interactions is the effect of the initial (residual) stress, which is almost always present in real structures and parts of machines, thus making development of effective methods of calculation of stress-strain state, taking into account the initial deformation, relevant and important scientific and technical challenge.

Today results on wide range of issues concerning contact problems for elastic, viscoelastic and plastic bodies have been obtained. These results are sufficiently presented in numerous periodical publications. Despite the significant achievements the number of research on contact interaction of bodies with initial stresses is relatively small. Detailed review of problems of contact interaction of elastic bodies with initial stresses is presented in [1 – 4].

The first works on the contact interaction of bodies with initial stresses describe interaction of pre-stressed rigid bodies with rigid and elastic punches without initial stress [1, 3, 5]. Thus either elastic potentials of concrete structures are considered, or the task relates in general to compressible (incompressible) bodies with the potential of arbitrary structure on the basis of linear elasticity theory. There is also a number of other generalizing publications that

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are partially or wholly related to the topic of this article [6 – 8].

This paper, using ratios of linear elasticity theory [4] presents a solution to the contact problem of elastic cylindrical punch with initial (residual) stresses pressing on the elastic layer and the base with initial (residual) stresses. The following cases are considered: the layer is positioned on a rigid base without friction; layer with initial stress is rigidly fixed to the undeformed (unstrained) base; layer with initial stress is positioned without friction on a base with initial stress; layer with initial stress is positioned without friction on a base without initial stress. Research is done in general for compressible and incompressible bodies for the theory of large initial deformations and two versions of small initial deformations theory with the random structure of elastic potential.

We believe that initial deflected modes in the layer, punch and the base are homogeneous and equal, and elastic potentials are twice continuously-differentiable functions of Green strain tensor algebraic invariants [4].

For research purposes Lagrange coordinates (x1, x2, x3) coinciding in their initial state with Cartesian coordinates (y1, y2, y3), with related ratios $y_i = \lambda_i x_i \quad (i = \overline{1,3})$ have been introduced. Punch, layer and base materials are considered isotropic compressible or incompressible. In the case of orthotropic materials it is assumed that elastic-equivalent directions coincide with the directions of the coordinate axes.

All values relating to the elastic cylindrical punch are marked with superscript "(1)", those referring to the layer and the foundation are denoted with upper indexes "(2)" and "(3)" respectively.

Problem setting and key ratios. We are going to distinguish three states of bodies with initial stresses: natural state, when all bodies have no stress; initial state and disturbed state, all values of which consist of the sum of the respective values of the initial state and disturbances. Considering disturbance to be much less than corresponding variables of original state, we do the research within linear elasticity theory [1 - 6, 9, 10].

Consider elastic cylindrical punch (Figure 1) with radius R and height H with the initial stress that is pressed into the elastic layer under the influence of force P after the initial strain state occurred. The thickness of the layer in its original deformed state is related to the thickness of its unstrained state by $h_1 = \lambda_3 h_2$ relation. We assume that the external load is applied only to the free end of the elastic punch, under which all points of punch move towards the axis of symmetry y_3 by the same value ε . We assume that the surface outside the contact area remains free from the influence of external forces, and no friction occurs in the contact zone while the movement and stress are continuous.

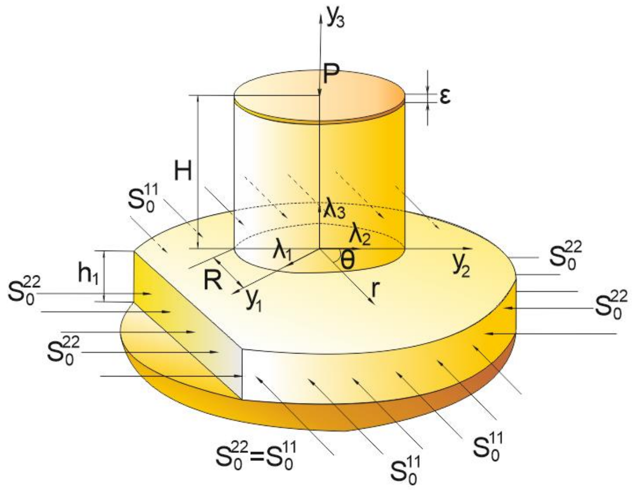

Figure 1. Pressure of a cylindrical punch on a layer and a foundation with initial (residual) stresses

Assume that the initial state of bodies is uniform and ratio [4] is executed: $y_m = x_m + U_m^0,\ U_m^0 = \delta_{mi}(\lambda_m - 1)\lambda_i^{-1} y_i$. $(i, m = 1,2,3)$

Then the basic equation in movements [8] for compressible bodies has a formula

$$L'_{m\alpha}U_\alpha = 0,\quad L'_{m\alpha} = \omega'_{ij\alpha\beta}\,\partial^2\big/\partial y_i \partial y_\beta\ ,\quad (i, m, \alpha, \beta = \overline{1,3}), \tag{1}$$

and for incompressible bodies together with incompressibility condition:

$$L'_{m\alpha}U_\alpha + q'_{\alpha m}\,\partial p'/\partial y_\alpha = 0,\ L'_{m\alpha} = \kappa'_{im\alpha\beta}\,\partial^2\big/\partial y_i \partial y_\beta\ , \tag{2}$$

$q'_{ij}\,\partial U_j\big/\partial y_i = 0,\ q'_{ij} = \lambda_i q_{ij}$, $(i, j, m, \alpha, \beta = \overline{1,3})$.

Expressions for determining the components of the stress tensor for compressible and incompressible bodies can be written as:

$$Q'_{ij} = \omega'_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial y_\beta},\ Q'_{ij} = \kappa_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial y_\beta} + q'_{ij}\,p,\ \omega'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\omega_{ij\alpha\beta}, \kappa'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\kappa_{ij\alpha\beta}\ ,$$

At the initial homogeneous stresses $S_0^{11} = S_0^{22} \neq 0;\ S_0^{33} = 0;\ \lambda_1 = \lambda_2 \neq \lambda_3$ solutions of equations (1) and (2) can be represented by cylindrical coordinates (r, θ, z_i) in the form of solutions of the equation:

$$(\Delta_1 + \xi_2'^2\,\partial^2\big/\partial\, y_3^2)(\Delta_1 + \xi_3'^2\,\partial^2\big/\partial\, y_3^2)\tilde{\chi} = 0 \tag{3}$$

where. $\Delta_1 = \partial^2\big/\partial r^2 + r^{-1}\,\partial/\partial r$

Given the condition of the existence of a single solution to linear elasticity theory for compressible and incompressible bodies [4], there are two options of presenting the total solution (3): in case of equal roots $(\xi_2'^2 = \xi_3'^2)$ [9] and a case of unequal roots $(\xi_2'^2 \neq \xi_3'^2)$ [10]. In this article we are considering the case of unequal roots of the equation (3), that is:

$$\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2,\ \left(\Delta_1 + \xi_2'^2\,\partial^2\big/\partial\, y_3^2\right)\tilde{\chi}_1 = 0,\ \left(\Delta_1 + \xi_3'^2\,\partial^2\big/\partial\, y_3^2\right)\tilde{\chi}_2 = 0\ . \tag{4}$$

In the circular cylindrical coordinate system $(r,\ \theta,\ z_i)$, where $z_i = v_i^{-1} y_3,\quad v_i = \sqrt{n_i},\ (i = \overline{1,\ 2})$, $n_1 = \xi_2'^2,\ n_2 = \xi_3'^2$, this formulation corresponds to the following boundary conditions:

1) at the end of an elastic punch:

$$u_3^{(1)} = -\varepsilon;\quad \tilde{Q}_{3r}^{(1)} = 0\quad (0 \leq r \leq R) \tag{5}$$

2) on the elastic layer boundary in the contact area []:

$$u_3^{(1)} = u_3^{(2)};\quad \tilde{Q}_{33}^{(1)} = \tilde{Q}_{33}^{(2)}\quad \tilde{Q}_{3r}^{(1)} = \tilde{Q}_{3r}^{(2)} = 0\quad (0 \leq r \leq R) \tag{6}$$

3) on the elastic layer boundary outside the contact area []:

$$\tilde{Q}_{33}^{(2)} = 0\quad \tilde{Q}_{3r}^{(2)} = 0\quad (R \leq r < \infty); \tag{7}$$

4) on the side surface of the elastic punch $r = R$:

$$\tilde{Q}_{rr}^{(1)} = 0;\quad \tilde{Q}_{3r}^{(1)} = 0\quad (0 \leq z_i \leq \frac{H}{v_i}). \tag{8}$$

At the bottom surface of the layer that lies on a rigid base and is secured with the base $z_i = -\dfrac{\lambda_3 h_2}{\nu_i} = -\dfrac{h_i}{\nu_i}, \; \left(i=\overline{1,2}\right)$,

$$u_3^{(2)} = 0 \quad \tilde{Q}_{3r}^{(2)} = 0 \quad (0 \le r < \infty)\,, \tag{9}$$

$$u_3^{(2)} = 0 \quad u_r^{(2)} = 0 \quad (0 \le r < \infty), \tag{10}$$

where $z_i = \dfrac{y_3}{\nu_i}, \quad \left(i=\overline{1,2}\right)$ the thickness of the layer in the unstrained state.

The equilibrium condition, which establishes the connection between the end subsidence and resultant load P has the form

$$P = -2\pi R^2 \int_0^1 \rho Q_{33}^{(2)}(0,\rho)\, d\rho. \tag{11}$$

To determine the stress-strain state of elastic cylinder in case of unequal roots general solution (4) of defining equation (3) takes the form:

$$\tilde{\chi} = 0,5\varepsilon\left\{\theta_8^{-1}(r^2 - z_1^2 - z_2^2) - \chi_0\left[r^2\left(\theta_8^{-1} + (2H\theta_6)^{-1}(z_1+z_2)\right) - \theta_8^{-1}(z_1^2+z_2^2) - (2H\theta_6)^{-1}(z_1^3+z_2^3)\right]\right\} - \tag{12}$$

$$-\sum_{k=1}^{\infty}\left\{b_3^{(k)}\left[s_0\frac{I_1(\gamma_k \nu_2 R)}{I_1(\gamma_k \nu_1 R)} I_0(\gamma_k \nu_1 r)\sin(\gamma_k z_1 \nu_1) + I_0(\gamma_k \nu_2 r)\sin(\gamma_k z_2 \nu_2)\right] - J_0(\alpha_k r)\left[\tilde{S}_2(\alpha_k z_1) + \tilde{S}_3(\alpha_k z_2)\right]\right\}\chi_k$$

where $s_0 = \dfrac{1+m_2}{1+m_1}$, $\theta_8 = m_1 n_1^{-1} + m_2 n_2^{-1}$, $\theta_6 = m_1 \nu_1^{-3} + m_2 \nu_2^{-3}$, $W(j) = \dfrac{(\tilde{c}_0 - \tilde{c}_j) I_0(\gamma_k \nu_j R)}{I_1(\gamma_k \nu_j R)} + \dfrac{1-\tilde{c}_0}{\gamma_k \nu_j R}$,

$$b_3^{(k)} = 4\varepsilon R^2 J_0(\mu_k)\left[\frac{\tilde{c}_1 - \tilde{c}_0}{\mu_k^2 + (\gamma_k \nu_1 R)^2} - \frac{\nu_2}{\nu_1 s_0}\frac{\tilde{c}_2 - \tilde{c}_0}{\mu_k^2 + (\gamma_k \nu_2 R)^2}\right](\nu_1 H \gamma_k^3 I_1(\gamma_k \nu_2 R)\left[\nu_2 W_k(2) - \nu_1 s_0 W_k(1)\right])^{-1},$$

$$\tilde{c}_0 = \begin{cases} \omega'_{1111}\omega'^{-1}_{1122}; \\ \lambda_1 q_1 (\lambda_3 q_3)^{-1}(\kappa'_{1133} + \kappa'_{1313})\kappa'^{-1}_{1122}; \end{cases} \quad \tilde{c}_i = \begin{cases} \lambda_3 \omega'_{1133} m_i \omega'^{-1}_{1122} n_i^{-1}; \\ (\kappa'_{1133} m_i - \kappa'_{3113})\kappa'^{-1}_{1122} n_i^{-1}; \end{cases} \quad (i=\overline{1,2}) .$$

Then expressions for the components of the movement vector and strain tensor for a cylindrical punch will look as:

$$U_r'^{(1)} = \varepsilon\theta_+ r(2H\theta_6)^{-1}\chi_0 + \sum_{k=1}^{\infty}\left\{\gamma_k^2 b_3^{(k)}\left[s_0 I_1(\gamma_k \nu_2 R)(I_1(\gamma_k \nu_1 R))^{-1}\nu_1 I_1(\nu_1 \gamma_k r)\cos(\gamma_k z_1 \nu_1) - \nu_2 I_1(\nu_2 \gamma_k r)\cos(\gamma_k z_2 \nu_2)\right] + \right.$$

$$\left. +\alpha_k^2 J_1(\alpha_k r)\left(\tilde{S}_4(\alpha_k z_1)\nu_1^{-1} + \tilde{S}_5(\alpha_k z_2)\nu_2^{-1}\right)\right\}\chi_k$$

$$U_3'^{(1)} = -\varepsilon\left\{1 + \chi_0\left[\frac{1}{H\theta_6}\left(\frac{m_1 z_1}{n_1} + \frac{m_2 z_2}{n_2}\right) - 1\right]\right\} - \sum_{k=1}^{\infty}\left\{\gamma_k^2 b_3^{(k)}\left[s_0\frac{I_1(\gamma_k \nu_2 R)}{I_1(\gamma_k \nu_1 R)} m_1 I_0(\gamma_k \nu_1 r)\sin(\gamma_k z_1 \nu_1) - m_2 I_0(\gamma_k \nu_2 r)\sin(\gamma_k z_2 \nu_2)\right] + \right.$$

$$\left. +\alpha_k^2 J_0(\alpha_k r)\left(\frac{m_1 \tilde{S}_2(\alpha_k z_1)}{n_1} + \frac{m_2 \tilde{S}_3(\alpha_k z_2)}{n_2}\right)\right\}\chi_k$$

$$Q_{33}^{\prime(1)}=C_{44}\left(1+m_1\right)l_1\left\langle-\frac{\varepsilon}{H\theta_6}\chi_0\left[\frac{1}{v_1}+\frac{s}{v_2}\right]-\right.$$

$$\left.-\sum_{k=1}^{\infty}\left\{\gamma_k^3 b_3^{(k)}\left[s_0\frac{I_1(\gamma_k v_2 R)}{I_1(\gamma_k v_1 R)}n_1 I_0(\gamma_k v_1 r)\cos(\gamma_k z_1 v_1)-sn_2 I_0(\gamma_k v_2 r)\cos(\gamma_k z_2 v_2)\right]+\alpha_k^3 J_0(\alpha_k r)\left(\frac{\tilde{S}_4\left(\alpha_k z_1\right)}{v_1}+\frac{s\tilde{S}_5\left(\alpha_k z_2\right)}{v_2}\right)\right\}\chi_k\right\rangle \qquad (13)$$

$$Q_{3r}^{\prime(1)}=C_{44}\left(1+m_1\right)\sum_{k=1}^{\infty}\left\{s_0\gamma_k^3 b_3^{(k)}\left[v_2 I_1(\gamma_k v_2 r)\sin(\gamma_k z_2 v_2)-v_1 I_1(\gamma_k v_2 R)(I_1(\gamma_k v_1 R))^{-1}I_1(\gamma_k v_1 r)\sin(\gamma_k z_1 v_1)\right]+\right.$$

$$\left.+\alpha_k^3 J_1(\alpha_k r)\left[n_1^{-1}\tilde{S}_2\left(\alpha_k z_1\right)+s_0 n_2^{-1}\tilde{S}_3\left(\alpha_k z_2\right)\right]\right\}\chi_k$$

where $\theta_+=v_1^{-1}+v_2^{-1}$,

$$\tilde{S}_2(\alpha_k z_1)=R^2\varepsilon\mu_k^{-2}\left[ch(\alpha_k z_1)-cth(\mu_k l v_1^{-1})sh(\alpha_k z_1)\right],\tilde{S}_4(\alpha_k z_1)=R^2\varepsilon\mu_k^{-2}\left[sh(\alpha_k z_1)-cth(\mu_k l v_1^{-1})ch(\alpha_k z_1)\right],$$

$$\tilde{S}_3(\alpha_k z_2)=\frac{n_2 R^2\varepsilon}{n_1\mu_k^2 s_0}\left[cth(\mu_k l v_2^{-1})sh(\alpha_k z_2)-ch(\alpha_k z_2)\right],\ \tilde{S}_5(\alpha_k z_2)=\frac{n_2 R^2\varepsilon}{n_1\mu_k^2 s_0}\left[cth(\mu_k l v_2^{-1})ch(\alpha_k z_2)-sh(\alpha_k z_2)\right].$$

Stressed-deformed state of elastic layer with initial (residual) stresses for unequal roots $n_1\neq n_2$ is determined from [10] by harmonic functions in the form of Hankel radial integrals. Having satisfied the third condition (6), the second one (7) and conditions (9), (10), after a number of changes we will have

$$u_r^{\prime(2)}=\hat{T}^3(\Omega_+^2;\ S_1^1;\ K_0^1;\ s_3;1;1)$$

$$u_3^{\prime(2)}=-m_1 v_1^{-1}\hat{T}^3(\Omega_-^2;\ S_1^0;\ K_0^0;\ s_3;\ s_2;1) \qquad (14)$$

$$Q_{33}^{\prime(2)}=C_{44}(1+m_1)l_1 R^{-1}\hat{T}^3(\Omega_+^2;\ S_2^0;\ K_1^0;\ s_3;\ s;1),$$
$$Q_{3r}^{\prime(2)}=C_{44}(1+m_1)s_3(v_1 R)^{-1}\hat{T}^3(\Omega_-^2;\ S_2^1;\ K_1^1;\ 1;\ 1;1)$$

where $s_1=\frac{m_1-1}{m_1}$, $s_2=\frac{m_2}{m_1}\frac{v_1}{v_2}$, $s_3=s_0\frac{v_1}{v_2}$, $s=s_0\frac{l_2}{l_1}$,

$$\hat{T}^3\left(\Omega_\pm^{n_1};\ S_{m_1}^n;\ K_{m_2}^n;\ \bar{\beta}_1;\ \bar{\beta}_2;l_1\right)=\frac{1}{\pi}\sum_{j=0}^{\infty}C_j^*\left[l_1\Omega_\pm^{n_1}\left(S_{j+m_1}^n;1;0;\ \bar{\beta}_1;\ \bar{\beta}_2;0\right)+\frac{1}{h}\sum_{i=1}^{\infty}a_i\Omega_\pm^{n_1}\left(S_{j+m_1}^n;h;0;\ \bar{\beta}_1;\ \bar{\beta}_2;k_i\right)\right]+$$

$$\frac{\varepsilon}{\pi\theta_3}(\chi_0-1)\left[l_1\Omega_\pm^{n_1}\left(S_{m_1}^n;1;0;\ \bar{\beta}_1;\ \bar{\beta}_2;0\right)+\sum_{i=1}^{\infty}a_i\Omega_\pm^{n_1}\left(S_{m_1}^n;h;0;\ \bar{\beta}_1;\ \bar{\beta}_2;k_i\right)\right]-$$

$$-\frac{\varepsilon\theta_4}{\pi\theta_3}\sum_{j=1}^{\infty}\chi_j\left[l_1\Omega_\pm^{n_1}\left(K_{m_2}^n;1;\mu_j;\ \bar{\beta}_1;\ \bar{\beta}_2;0\right)+\frac{1}{h}\sum_{i=1}^{\infty}a_i\Omega_\pm^{n_1}\left(K_{m_2}^n;h;\mu_j;\ \bar{\beta}_1;\ \bar{\beta}_2;k_i\right)\right]$$

$$\Omega_\pm^2\left(\hat{L}_m^n;t;\mu;k;a;\theta\right)=k\left[\hat{L}_m^n\left(\frac{\rho}{t},\mu,-\frac{z_1}{tR}+\theta\right)\pm\hat{L}_m^n\left(\frac{\rho}{t},\mu,\frac{z_1}{tR}+\frac{h_1}{tRv_1}+\theta\right)+\hat{L}_m^n\left(\frac{\rho}{t},\mu,-\frac{z_1}{tR}-\frac{h_1}{tRv_1}+\theta\right)\right]-$$

$$-\mathrm{a}\left[\hat{L}_m^n\left(\frac{\rho}{t},\mu,\frac{z_2}{tR}+\theta\right)\pm\hat{L}_m^n\left(\frac{\rho}{t},\mu,\frac{z_2}{tR}+\frac{h_1}{tRv_2}+\theta\right)+\hat{L}_m^n\left(\frac{\rho}{t},\mu,-\frac{z_2}{tR}-\frac{h_1}{tRv_2}+\theta\right)\right],\ \hat{L}_m^n(t,0,u)=\hat{L}_m^n(t,u).$$

$S_1^1(\rho;z)=\frac{\sqrt{2}}{2\rho}(\sqrt{\sqrt{(\rho^2+z^2-1)^2+4z^2}-2z}+\sqrt{2}),\quad S_1^0(\rho;z)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}\Gamma(4k-2)\rho^{4k-3}}{2^{2k-1}(2k-1)!\Gamma(2k)(z^2-1)^{4k-2}},$

$S_2^0(\rho;z)=-\frac{\sqrt{2}}{2}\sqrt{\frac{\sqrt{(\rho^2+z^2-1)^2+4z^2}-2z}{(\rho^2+z^2-1)^2+4z^2}},\quad S_2^1(\rho;z)=\frac{1}{\rho}\sqrt{\frac{\sqrt{(\rho^2+z^2-1)^2+4z^2}-2z}{(\rho^2+z^2-1)^2+4z^2}},$

$K_n^m(\rho;\mu_k;z)=\int_0^{\infty}\eta^n\psi_0(\eta,\mu_k)e^{z\cdot\eta}J_m(\eta\rho)d\eta$, k_i, a_i – certain constants (i=0,1,2,...).

In (12) – (14) coefficients n_i, m_i, c_{44}, l_i are given in [4].

The method of solution. Using the solutions for cylinder (12), (13) and satisfying the third condition (6) and the second condition (8), we find the eigenvalues of the problem (5) - (11) in the case of unequal roots $n_1 \neq n_2$

$$\gamma_k=\frac{\pi(2k+1)}{H},\quad \alpha_k=\frac{\mu_k}{R},\ \text{where}\ J_1(\mu_k)=0. \tag{15}$$

With the first conditions (6) and (7) we can determine the unknown function $F(\eta)$ of dual integral equations for unequal roots

$$\int_0^{\infty}F(\eta)\eta^{-1}J_0(\eta\rho)d\eta=f(\rho),\ (\rho<1),\quad \int_0^{\infty}F(\eta)J_0(\eta\rho)d\eta=0,\ (\rho>1), \tag{16}$$

where $f(\rho)=\frac{\varepsilon}{\theta_3}(\chi_0-1-\theta_4\sum_{k=1}^{\infty}\chi_k J_0(\mu_k\rho)+\frac{\theta_3}{\varepsilon}\int_0^{\infty}\frac{F(\eta)}{\eta}G(\eta h)J_1(\eta\rho)\,d\eta),\quad \theta_4=\frac{v_1(m_2-1)-m_1s_0}{n_1},\quad \theta_3=\frac{m_1}{v_1}(s_1-s_0),$

Applying reciprocation formula to (16) leads to a Fredholm integral equation of the second kind regarding function $F(\eta)$

$$\frac{F(\eta)}{\eta}=\frac{2\varepsilon}{\pi\theta_3}\left((\chi_0-1)\psi_0(\eta,0)-\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\eta,\mu_k)+\frac{\theta_3}{\varepsilon}\int_0^{\infty}\frac{F(u)}{u}G(uh)\psi_0(\eta,u)du\right), \tag{17}$$

where. $\psi_n(x,y)=\int_0^1 t^n\cos xt\cos yt\,dt$.

Satisfying the second boundary condition (6), we will look for solution (17) using a method of successive approximations, based on zero approach function

$F^{(0)}(\eta)/\eta=2\varepsilon(\pi\theta_3)^{-1}p(\eta),$

where $p(\eta)=(\chi_0-1)\psi_0(\eta,0)-\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\mu_k\eta).$

The following approaches can be determined by the formula

$\frac{F^{(j)}(\eta)}{\eta}=\frac{2}{\pi}\int_0^{\infty}\frac{F^{(j-1)}(u)}{u}G(uh)J_0(\eta u)du$

Solution (14) is written as

$$F(\eta)=\sum_{n=0}^{\infty}F^{(n)}(\eta). \tag{18}$$

Note that the process of successive approximations (18) is convergent at $h>1$, but due to the bulkiness its proof is not presented here.

Satisfying the first two boundary conditions (6) given orthogonality of Bessel functions $J_0(\mu_k\rho)$ to determine constants χ_i $(i=0,1,2,..)$ we obtain an infinite quasi-regular system of algebraic equations

$$\vartheta_k\chi_k+\sum_{n=0}^{\infty}\vartheta_{kn}\chi_n=\varpi_k \quad (k=0,1,2,..). \tag{19}$$

The coefficients of the system can be represented as

$$\vartheta_0=\varpi_0=\frac{2}{\pi}\left[1+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{\infty}\frac{\sin u}{u}G(hu)\psi_{j-1}(u,0)du\right];\quad \vartheta_{0n}=\frac{2}{\pi}\left[-\theta_4\psi_0(0,\mu_n)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{\infty}\frac{\sin u}{u}G(hu)\psi_{j-1}(u,\mu_n)du\right];$$

$$\vartheta_{k0}=\frac{2}{\pi}\left[-\theta_4\psi_0(0,\mu_k)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{\infty}\left(\psi_{j-1}(u,\mu_k)G(hu)\psi_0(u,0)du\right);\quad \vartheta_{00}=\frac{\theta_5\theta_3RE}{\kappa l};\right. \tag{20}$$

$$\vartheta_k=\frac{\theta_3\mu_kJ_0^2(\mu_k)}{2\kappa Rv_1}\left[\frac{l_2v_2}{l_1v_1}cth\left(\frac{\mu_kl}{v_2}\right)-cth\left(\frac{\mu_kl}{v_1}\right)\right];\quad \varpi_k=\frac{2}{\pi}\left[\psi_0(0,\mu_k)+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{\infty}\psi_{j-1}(u,\mu_k)G(hu)\psi_0(u,0)du\right];$$

$$\vartheta_{kn}=\frac{2}{\pi}\left[-\theta_4\psi_0(\mu_k,\mu_n)-\frac{2\theta_3s_0v_1R\pi}{\kappa l}\sum_{m=1}^{\infty}\tau_{mn}\iota_{km}+\frac{2}{\pi}\sum_{j=1}^{\infty}\int_0^{\infty}\left(\psi_0(u,\mu_n)G(hu)\psi_{j-1}(u,\mu_k)du\right)\right];$$

де $\theta_5=(v_2+v_1s)n_1n_2((m_1v_2^3+m_2v_1^3)E)^{-1}$, $\psi_j(\eta,\mu_n)=\frac{2}{\pi}\eta\int_0^1\cos\eta tdt\int_0^{\infty}\frac{\psi_{j-1}(u,\mu_n)}{u}G(uh)\cos utdu.$

In calculating functions (18) and the coefficients (20), most integrals in the final form is not calculated, given the complexity of the functions G_i $(i=\overline{1,4})$. Therefore, starting with the second approach, the integrands are factorized in series on degrees h^{-i}, (i=1,..,7) which allows to approximately calculate the system coefficients (19).

On defining the unknown constants χ_i $(i=0,1,2,..)$ out of the system (19), we can calculate the deflected mode in both the elastic punch and the layer using the formulas (13) – (14).

As a result, the solution is presented as a series through infinite system of constants determined from a system of quasi-regular linear equations. Moreover, in the system (19) coefficients ϑ_k and ϑ_{kn} depend on the structure of the elastic potential, elastic punch height H and thickness of the pre-stressed layer, whereas free members depend on the roots n_1, n_2.

Given the asymptotic representation for the Bessel functions, variables μk and limited integrals ψ (μk, μρ), the system (19) is quasi-regular if $\lambda_1>\lambda_{кр}$, as well as at the fulfillment of condition

$$C_{44}l_1(1+m_1)(s-s_0)(m_1(s_0-s_1))^{-1}<\begin{cases}0,36E(1-v^2)^{-1}, & \text{для стисливих тіл;}\\ 0,48E, & \text{для нестисливих тіл,}\end{cases}$$ for compressible bodies

for incompressible bodies

Numerical analysis. Numerically, quasi-regularity of the system (19) is confirmed by Table 1 which is formed for the first eight values of system coefficients, presented in the form of

$$\chi_k=-\sum_{n=0}^{\infty}\vartheta_{kn}/\vartheta_k\cdot\chi_n+\varpi_k/\vartheta_k \quad (k=0,1,2,..)$$

in the case of Treloar potential at h=4, λ_1=0.7.

Table № 1

Coefficients of quasi-regular system of linear algebraic equations

n/k	$-\vartheta_{kn}/\vartheta_k$								ϖ_k/ϑ_k
	1	2	3	4	5	6	7	8	
1	0.678172	0.678171	-0.49398	-0.49397	0.368706	0.368705	-0.27927	-0.27926	$3.62\cdot10^{-5}$
2	0.678139	0.678138	-0.49393	-0.49392	0.368649	0.368648	0.279211	0.279210	$-3.39\cdot10^{-6}$
3	0.678129	0.678128	-0.49392	-0.49391	0.368635	0.368634	-0.27920	-0.27919	$-1.26\cdot10^{-6}$
4	0.678095	0.678094	-0.49389	-0.49388	0.368611	0.368602	-0.27918	-0.27916	$-1.09\cdot10^{-6}$
5	0.678082	0.678081	-0.49384	-0.49383	0.368539	0.368538	-0.27911	-0.27910	$6.72\cdot10^{-7}$
6	0.678080	0.678079	-0.49383	-0.49382	0.368538	0.368537	-0.27910	-0.27909	$7.58\cdot10^{-8}$
7	0.678068	0.678067	-0.49382	-0.49381	0.368515	0.368514	-0.27908	-0.27907	$2.43\cdot10^{-8}$
8	0.677890	0.677889	-0.49352	-0.49351	0.368184	0.368183	-0.27875	-0.27874	$-8.97\cdot10^{-9}$

The paper also provides numerical solution of system (19) for Treloar potential (neohuk bodies) at these parameter values: k=n=32; $\nu=\nu_1=0,5$; l=10; $\lambda_1=$0,7; 0,8; 0,9; 1; 1,1; 1,2; E=3,92. The algorithm is based on the method of reduction and is implemented as a Maple package program.

Fig. 2 and 3 show the distribution of contact stresses under the punch $\frac{\pi R^2}{P}\tilde{Q}_{33}$ where values λ_1 correspond to the line, starting from the bottom to the top.

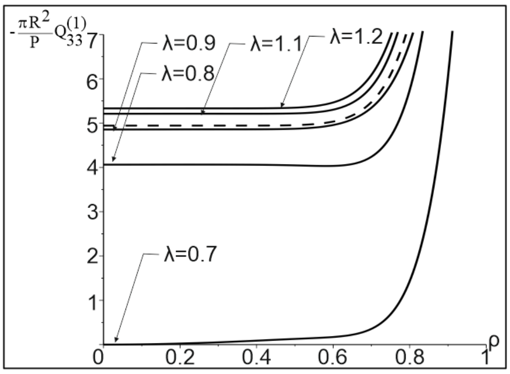

Figure 2. Distribution of contact stresses at h=1.6

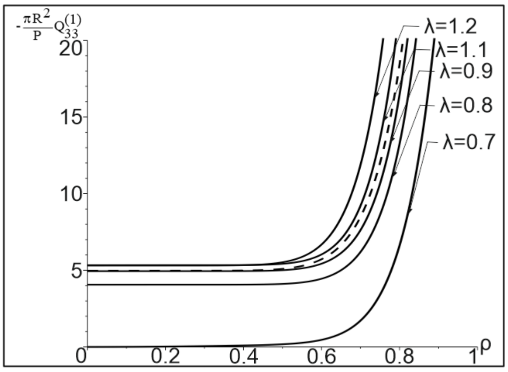

Figure 3. Distribution of contact stresses at h=4

In figures 2, 3 dotted lines describe the case without initial stress ($\lambda_1=1$), and solid lines describe the case with initial (residual) stresses.

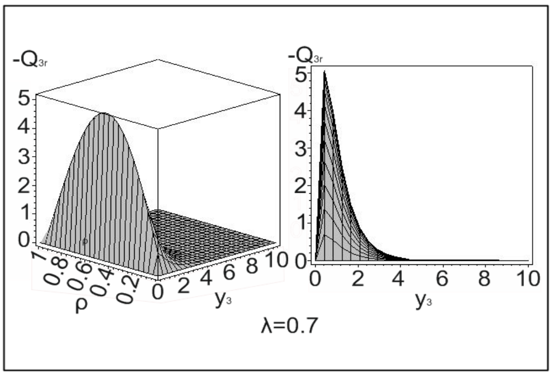

Figure 4. Tangential stress at h=1.6

Fig. 4 shows that tangential stress is mostly concentrated near the contact area.

A study of convergence of numerical series, which occurred during solving the problem (5) – (11), was conducted. Thus majorants were found for most series. The convergence of some series was quite difficult to prove analytically, but the numerical results showed that it is provided by monotonous descending of constants $\chi_i \quad (i = 0,1,2,..)$ and $|J_0(\mu_k \rho)|$. Some numerical series included in the expressions for the stress of a cylindrical punch (13) at the points of change of boundary conditions turned out to be divergent (since $\mu_k \cdot \chi_k \cdot J_0(\mu_k \rho) \to \infty$ when $k \to \infty$), but this is consistent with the research [4].

Conclusions. For potentials corresponding to unequal roots $n_1 \neq n_2$, when $\lambda_1 = 1$, the solution presented in the article under the linear elasticity theory considering initial stress is different from a similar solution for linear transversely isotropic body (without initial stress) because their corresponding overall solutions do not match.

Effect of initial stress on the deflected mode of elastic cylinder which is pressed into the elastic layer and the foundation is as follows: in the case of compression the original strains in a layer lead to reduction of stress in an elastic punch, whereas in case of tension they lead to their increase, and in case of movement the effects are opposite.

That is, the presence of pre-stressed state during contact interaction of elastic bodies makes it possible to adjust the contact stress and movements at structure durability calculations. Thus for contact stresses initial tensions are dangerous in case of stretching, and for movements initial stresses are dangerous in case of compression.

Comparing Figure 2 and Figure 3, which shows the contact stresses for a cylinder with initial stresses, it is obvious that the layer thickness does not affect the nature of the initial stress, and only affects their value.

Initial (residual) stresses have far greater quantitative impact on highly elastic materials compared to more rigid materials.

Discovered mechanical effect similar to earlier studies [1, 3, 4], which is that in the case when λ_1, approximates the values of material surface instability, phenomena of resonance character occur both in the layer and in the punch. They lie in the fact that the tension and movement of bodies that interact change their values considerably.

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УДК 539.3

ДОСЛІДЖЕННЯ КОНТАКТНОЇ ВЗАЄМОДІЇ ПОПЕРЕДНЬО НАПРУЖЕНИХ ШТАМПА, ШАРУ ТА ОСНОВИ БЕЗ УРАХУВАННЯ СИЛИ ТЕРТЯ

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Резюме. *У рамках лінеаризованої теорії пружності представлено розв’язок контактної задачі про тиск попередньо напружених циліндра, шару та основи без урахування сили тертя. Дослідження представлені в загальному вигляді для теорії великих початкових деформацій та двох варіантів теорії малих початкових деформацій при довільній структурі пружного потенціалу.*

Ключові слова: *лінеаризована теорія пружності, початкові (залишкові) напруження, метод послідовних наближень, метод редукції, інтегральні рівняння типу Фредгольма.*

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